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# IMPEDANCE AND CURRENT OF INSULATED AND END-GROUNDED LINEAR ANTENNAS IN SEAWATER

Giorgio V. Borgiotti

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This paper provides a comprehensive account of analytical results for computing the antenna impedance and the equivalent length of a linear antenna, either insulated or end-grounded, immersed in a conducting medium. The theoretical results and the concomitant computer codes are directly applicable to the analysis of the sensitivity and thermal noise of a submerged cable antenna.

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## FOREWORD

This paper presents the results of an electromagnetic analysis of submerged antennas. The work is part of an effort by the Institute for Defense Analyses (IDA) to support a joint DARPA-Navy design team. The team was formed to investigate the feasibility of invisible VLF receive antennas for SSBNs, and to develop a paper design for a selected antenna configuration.

One of the objectives of the Submarine Antenna Study has been to reach an understanding on whether a cable antenna, properly designed, can be submerged several meters deep to achieve radar and optical undetectability. To accomplish this objective, an accurate analytical tool has been developed to evaluate the dependence of the antenna sensitivity and thermal noise--related to the antenna resistance--upon the antenna configuration and physical parameters. This paper documents and discusses in detail this analytical tool. Numerical results for the impedance of a potentially low noise configuration are also presented. They have been instrumental in identifying the key design parameters and in establishing criteria for their selection. A comprehensive system evaluation of different antenna designs is, however, outside the scope of this report and is the subject of a separate paper.

## ACKNOWLEDGMENT

The Fortran codes for the numerical simulations were skillfully written by E. Marcuse.

## ABSTRACT

This paper provides a comprehensive account of analytical results for computing the antenna impedance and the equivalent length of a linear antenna, either insulated or end-grounded, immersed in a conducting medium. The theoretical results and the concomitant computer codes are directly applicable to the analysis of the sensitivity and thermal noise of a submerged cable antenna.

*The purpose of computing is insight, not numbers.*

*.... R.W. Hamming*

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## SUMMARY AND CONCLUSIONS

A computer model for a linear antenna in an unbounded, possibly dissipative medium has been developed, with the objective of providing a tool for the numerical evaluation of the antenna impedance and sensitivity. The model is flexible, its range of applications covers a variety of physical situations and antenna configurations. Among them, of great practical interest, are the insulated antenna in a conducting medium, the bare antenna in a conducting medium, and the antenna in partial contact with the medium via electrodes in the antenna terminal regions. The model applies also to antennas in a semi-infinite medium (submerged antennas in the ocean), provided the distance from the surface is greater than several antenna skin depths in the dissipative medium.

The analytical approach is based on an expression for the antenna impedance which is stationary with respect to the functional form of the antenna current, in the sense of being insensitive to a first-order variation of the current functional form with respect to the correct one. A drastic simplification of the analysis and of the numerical computations has been achieved by representing the field as a superposition of cylindrical waves, whose complex amplitudes--the Fourier Transforms of the field components in the direction of the antenna axis--depend in a relatively simple way upon the antenna current distribution.

Confirming the physical intuition, the numerical computations show that a desirable configuration for a VLF antenna submerged in sea water consists of an end-grounded cable, with

electrodes having a large area, and consequently a low contact resistance. Low impedance, low Q, and high sensitivity are desirable features of this design approach.

# 1. INTRODUCTION

## 1.1 MOTIVATION

The primary parameters characterizing the performance of an electrically small antenna in a communication system are two:

a) The antenna (open circuit) equivalent length.*

b) The antenna impedance.

It is the purpose of this paper to introduce and discuss in detail an analytical tool for the numerical evaluation of a) and b) for a thin cylindrical antenna in an unbounded, generally dissipative medium. The analytical model here presented is flexible and can accommodate different physical situations and antenna condigurations. Cases of particular importance are: the insulated antenna, the bare antenna, and the "grounded" cable antenna with the terminal regions in contact with the dissipative medium.

Consider a receive linear antenna of length h immersed in a field whose electric component in the direction of the antenna axis at the abscissa s along the antenna is E(s). Suppose now that the antenna is operating in transmission: further, assume that a current generator of unit strength feeds the antenna terminals. (This condition does not need to correspond to a practical or useful way of operation but rather should be considered as a conceptual device to evaluate certain quantities characterizing the antenna receive properties.)** Suppose

---

*The antenna gain is a concept of limited or no value for antennas in dissipative media.

**Recall that submarine VLF antennas are used only for reception.

the current distribution in transmit operation is I(s). Then, on the basis of reciprocity theorem [Ref. 1], it can be shown that the open circuit receive voltage is equal to

$$V_r = \int_0^h E(s)\, I(s)\, ds \qquad (1)$$

(Appendix A). If E(s) is a plane wave field, an effective length $\ell_e$ can be introduced, related to $V_r$ by

$$V_r = \ell_e E, \qquad (2)$$

with

$$\ell_e = \int_0^h I(s)\, ds\,. \qquad (3)$$

If the antenna input impedance is Z = R + jX, the open circuit thermal noise at the antenna open terminals is

$$P_N = 4KTR, \qquad (4)$$

where T is the temperature of both the antenna and the surrounding medium.* Equations (1) and (4) are general and apply to an antenna in an arbitrary medium, either lossless or dissipative. Clearly, their application requires the analysis of the antenna

*The antenna impedance in conjunction with the antenna Q, determines the overall noise figure of the antenna/front-end amplifier combination. If the antenna Q, defined as the ratio between the antenna reactance and resistance at center frequency, is low or moderate, the noise figure can be made low by tuning the reactive component of the antenna impedance [Ref. 2]. If the antenna Q is very high, see examples in Section 1.2, the loss in the tuning circuit and the input resistance of the front end amplifier will have the effect of increasing the system noise (and, of course, reducing the overall Q of the input stage/antenna combination). The discussion of these important issues is outside the scope of this report.

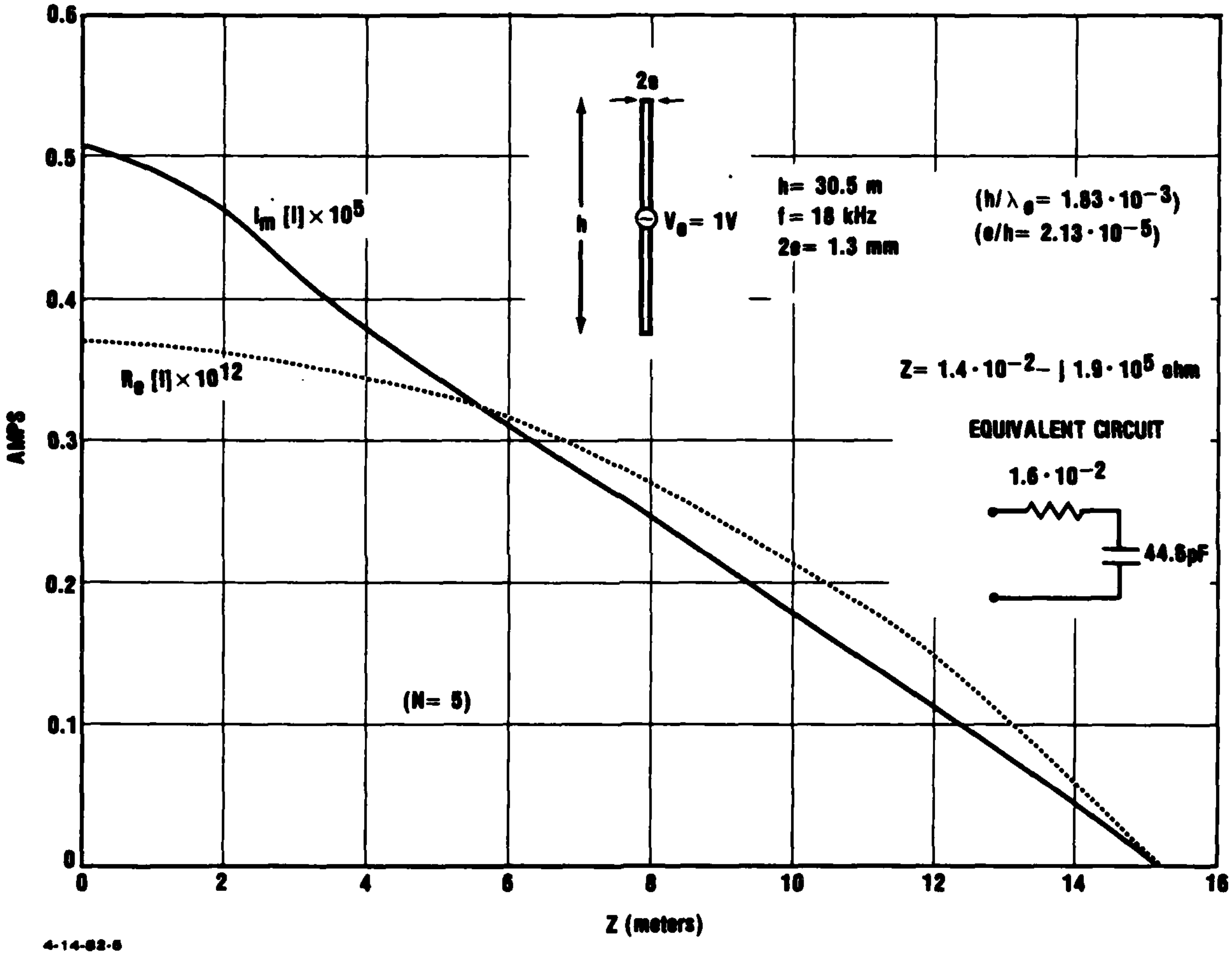


FIGURE 1.  Short dipole in free space

in transmit operation, with the objective of determining the antenna current distribution and input impedance.

## 1.2 BACKGROUND

An antenna in a conducting medium is a physical object very different from an antenna in free space, the difference stemming, of course, from the fact that the medium surrounding the antenna can support a conduction current in one case and not in the other. This has, in turn, an enormous effect on the antenna impedance and current distribution. To elaborate, consider Figs. 1 to 6. Figure 1 represents a short dipole in free space whose geometric parameters are

$$\frac{h}{e} = 4.69 \cdot 10^4 \quad ; \quad \frac{h}{\lambda_o} = 1.83 \cdot 10^{-3}$$

where $\lambda_o$ is the wavelength in free space. If the frequency were equal to 18 kHz, h would be equal to 30.5 m. The antenna impedance, calculated by using the computer model described later in this paper, is

$$Z = 1.4 \cdot 10^{-2} - j1.98\ 10^5 \text{ ohm}$$

and the equivalent circuit--valid for frequencies such that $h \ll \lambda_o$--consists of a small resistance in series with a large reactance corresponding to a very small series capacitance. As is well known, the current distribution along the antenna is approximately triangular, see Fig. 1, the current leading the excitation voltage by almost exactly 90 degrees.* This example

*To save computer time, only a small number of harmonics (N = 5) has been used here and in the example of Fig. 2 to represent the current (see Section 2). If a higher number of harmonics had been used, the triangular shape of the current distribution would be more clearly apparent. The impedance, however, is virtually unaffected.

shows clearly why such a short dipole is a most inefficient antenna. Consider, for the sake of discussion, the dipole in transmit operation. Tuning it with a lossless inductance, a physical impossibility, would result in a resonant circuit having a Q greater than $10^7$, with a concomitant 3 dB bandwidth of about 0.001 Hz for a center frequency equal to 18 kHz. Introucing losses in the tuning circuit will reduce the system Q to an acceptable value, but at the expense of the radiation efficiency, which will become a small fraction of unity. Parallel reasoning applies to the receive situation, for which it is easy to see that an enormously impractical high input impedance for the receiver front end would be necessary.

Suppose now that the same dipole is coated with an insulating jacket and immersed in sea water (Fig. 2). The jacket is assumed to have an outer diameter 2p = 16.5 mm with a relative dielectric constant $\varepsilon_p = 1.65$. The calculated impedance is

$$Z = 2.2\ 10^{-1} - j\ 3.32 \cdot 10^4 \text{ ohm}$$

and the equivalent circuit and current distribution are also shown in Fig. 2. The resistance has increased but it is still comparatively small with respect to the reactance. The system Q, still exceedingly high, is less than for the antenna in free space. The physical reason is, of course, that in the region surrounding the antenna the displacement current has been replaced by a conduction current, the reactance corresponding, heuristically, to the capacitive coupling between the antenna and the conducting medium. The current distribution is again triangular, with the current essentially in quadrature with the gap voltage. As expected, it is numerically found that, if the thickness of the insulating jacket is reduced, the capacitance between the antenna conductor and the salt water increases, with a reduction of the antenna impedance (see Fig. 3).

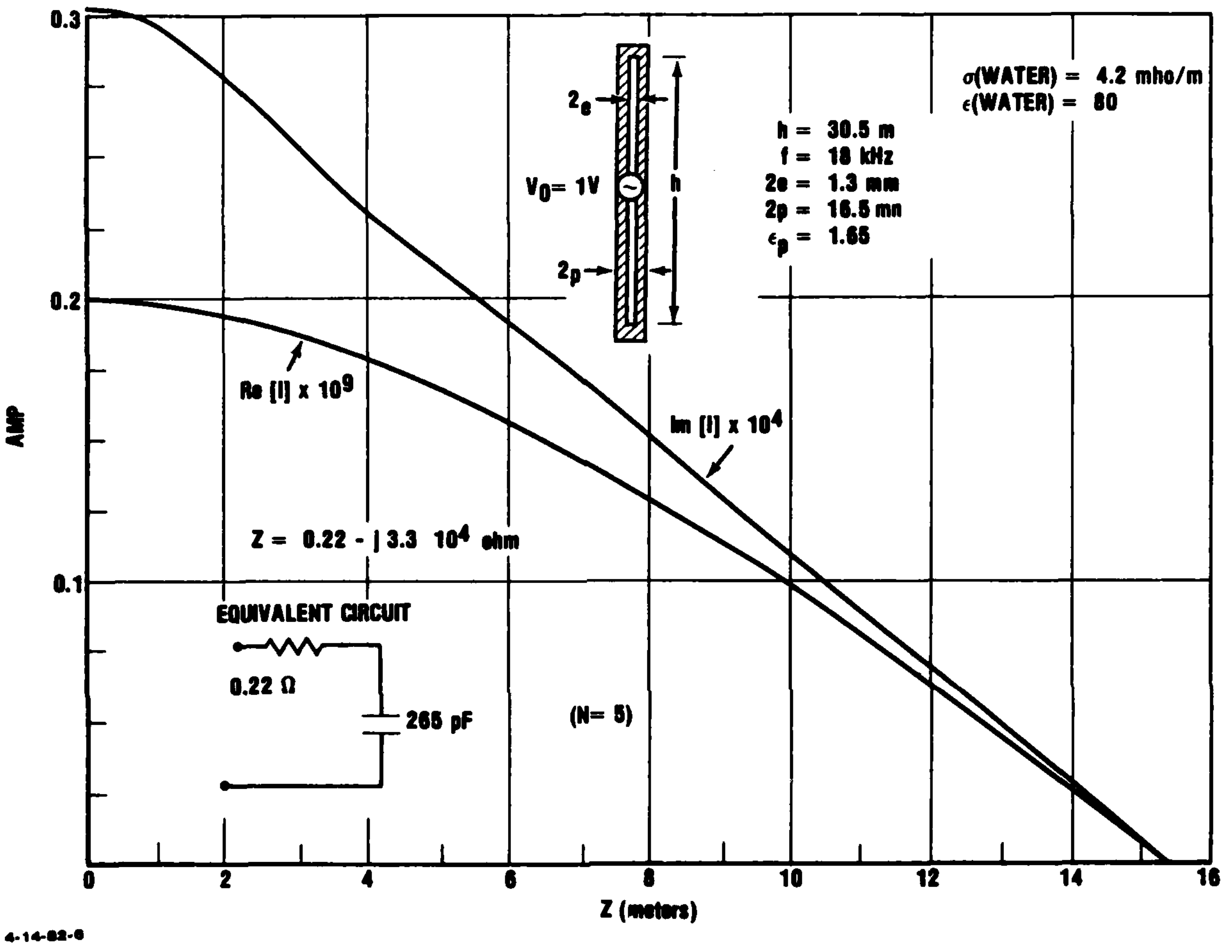


FIGURE 2. Short insulated dipole in seawater, example I

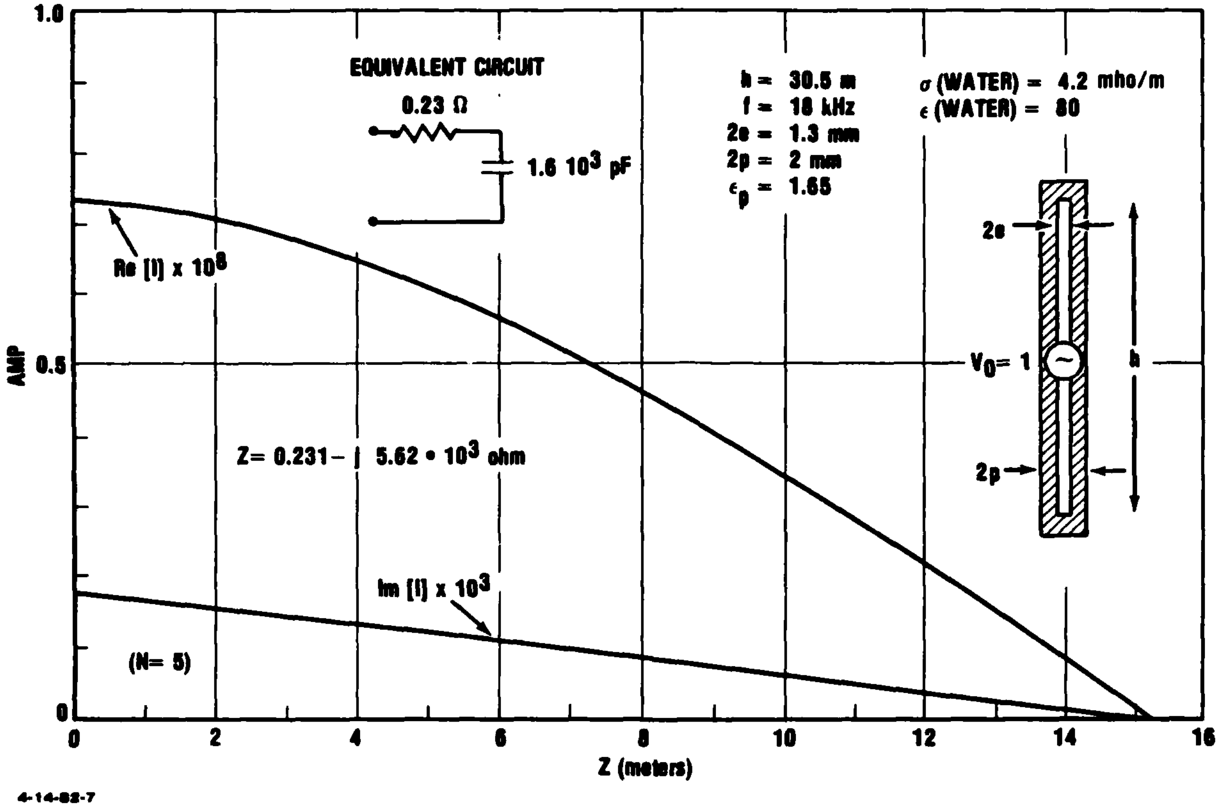


FIGURE 3. Short insulated dipole in seawater, example II

In Figs. 4 and 5 the results of the numerical analysis are shown for a completely different physical situation. The antenna is now "bare" and immersed in salt water. For the same frequency diameter and length of the antenna conductor as in the previous examples, the antenna impedance is

$$Z = 0.326 + j.353 \text{ ohm},$$

with a Q close to unity. It can be shown numerically that the impedance becomes insensitive to the antenna length, provided the latter is greater than a few skin depths in the medium. The reason is, of course, the rapid attenuation of the current along

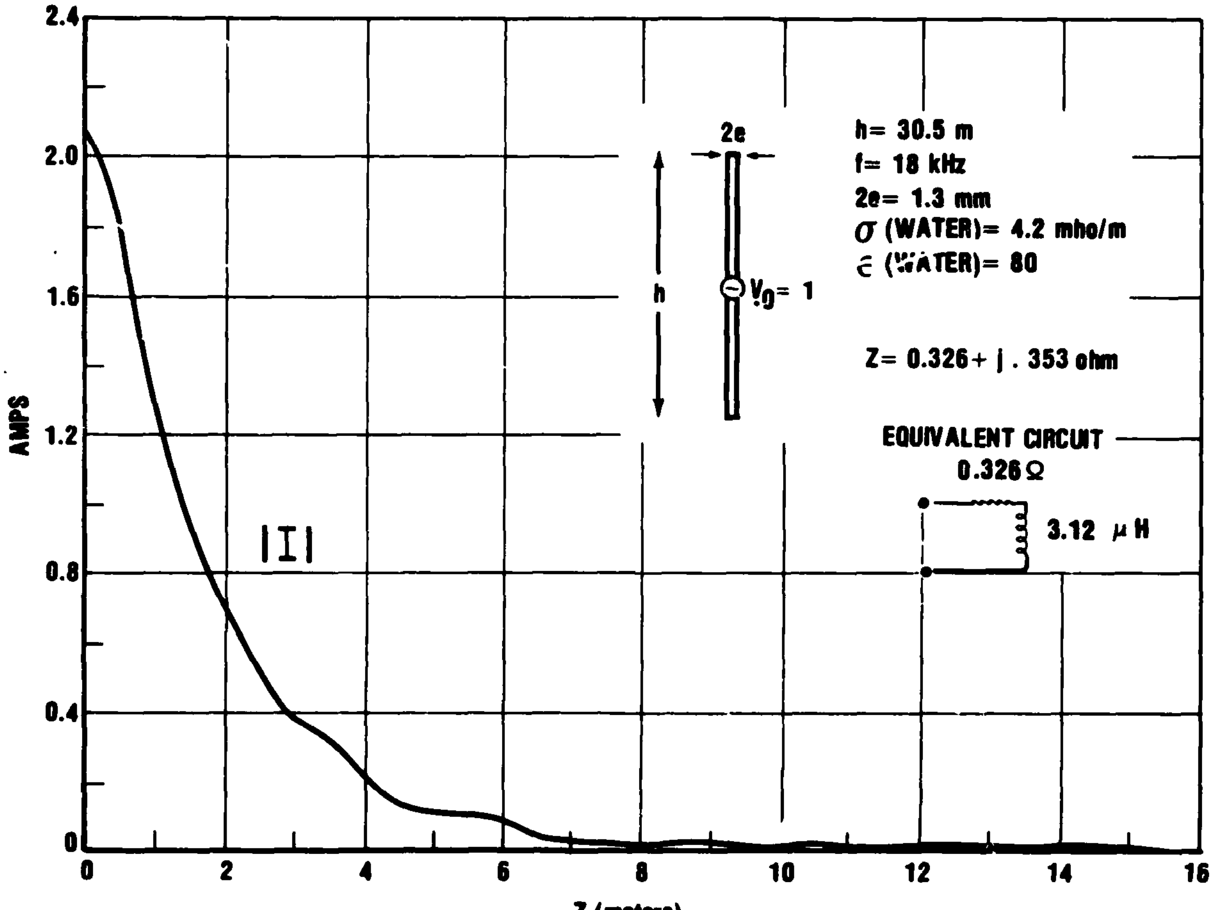


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FIGURE 4. "Bare" antenna in seawater modulus of the current

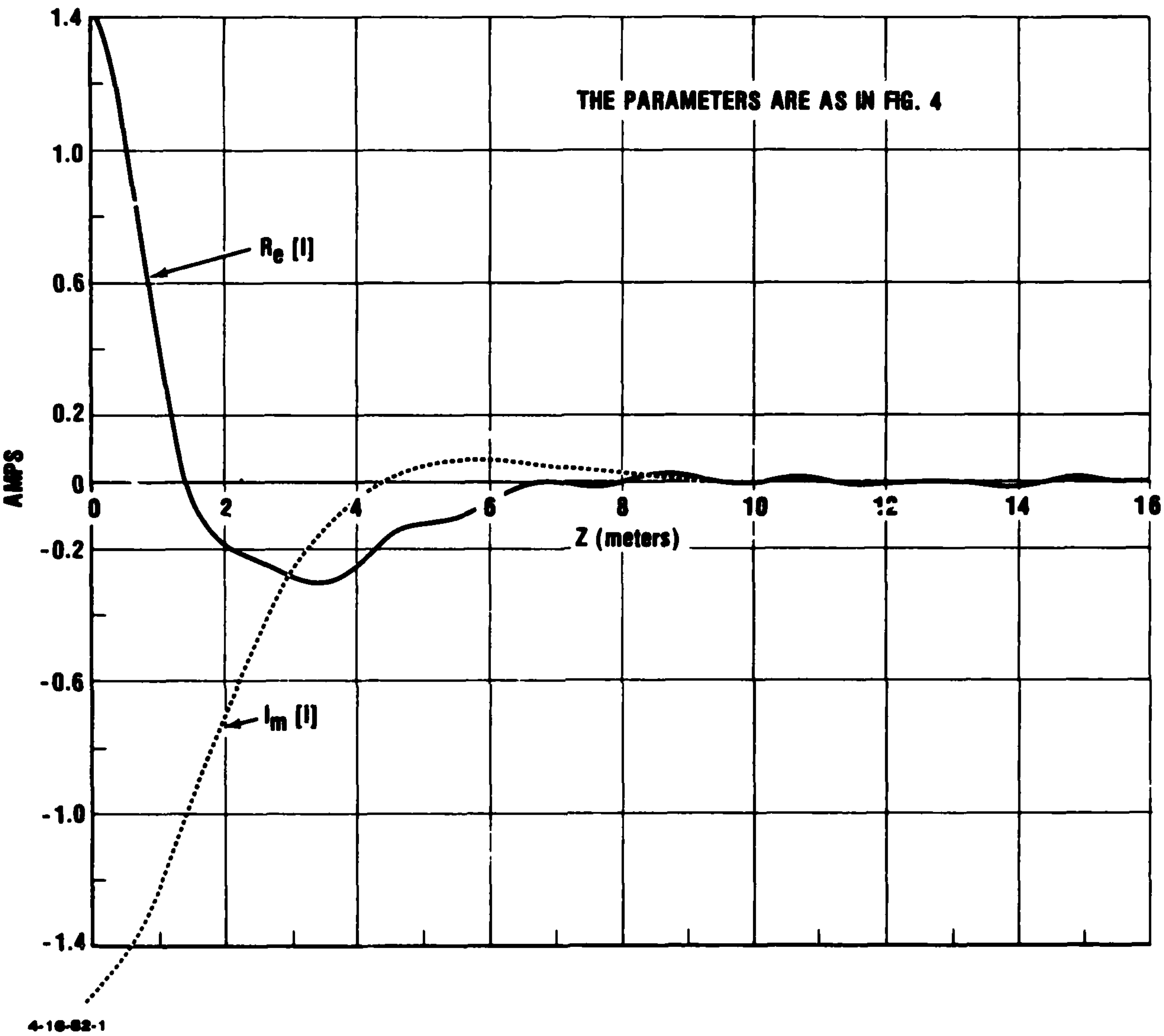


FIGURE 5. "Bare" antenna in salt water

the antenna with the distance from the feed point. In fact, except in the immediate vicinity of the feed point, the calculated antenna current distribution fits very well a decreasing exponential with an attenuation close to that of a plane wave in salt water, 4.66 dB/m*. Thus, in this kind of design, the antenna "active" parts is limited to a length equal to only several skin depths in salt water, no matter what the physical length of the antenna is. The antenna sensitivity is small. Clearly, this kind of design does not lead to a desirable antenna configuration.

Consider, finally, an antenna in a dissipative medium--sea water--insulated by a dielectric jacket, except at the ends where two cylindrical metal segments, the "electrodes", are in contact with the medium, Fig. 6. In this case, we heuristically expect the impedance to be much smaller than in the cases of Figs. 2 and 3, a very desirable feature. In addition, however, unlike the case of Figs. 4 and 5, the current distribution does not decay fast with the distance from the feeding point, being for $h \ll \lambda_o$ almost a constant (Fig. 6). Therefore, the antenna sensitivity is excellent, its effective length being practically equal to its physical length. From the heuristic discussion and the numerical examples in this section, and from the detailed analysis in the next sections, it is apparent that the structure of Fig. 6, conductively coupled to the medium, is a physical object very different in many respects from that of Figs. 2 and 3. For example, the antenna of Fig. 6 can be operated at or close to dc, unlike the insulated antenna of Fig. 2 (see Section 5). This physical difference has its mathematical counterpart in the somewhat different analytical formulation, although based on the same general concept, and in the concomitant computer code.

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*The fit to a decaying exponential would be even better if a larger number of harmonics had been used in representing the current when applying moment method.

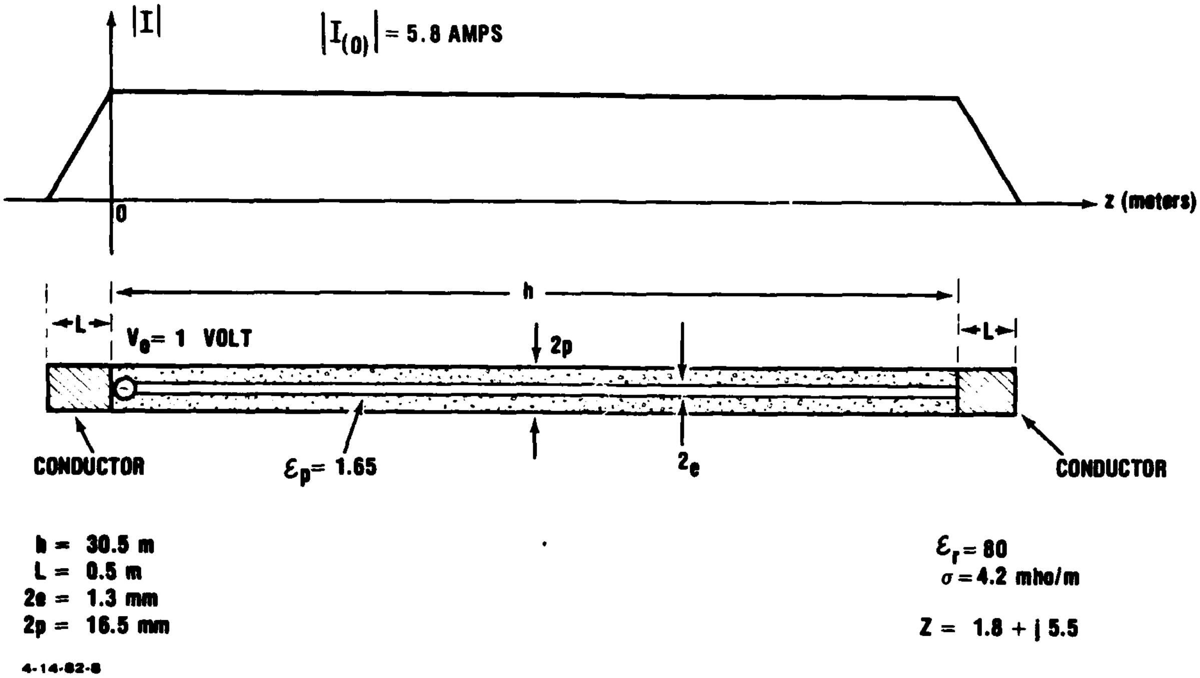


FIGURE 6. End-grounded antenna in seawater

## 1.3 ANALYTICAL APPROACH

The literature on linear antennas is, of course, immense. Yet it does not appear that a sufficiently general method of analysis for dipoles in a dissipative medium has hitherto been established. R.W.P. King and his coworkers, in a series of papers, addressed the problem of determining the impedance and the current of an insulated antenna in a lossy medium. Reference [3] is a concise tutorial introduction to the subject. The approach in Refs. [4] and [5], is based on an extension of Hallen's integral equation for the antenna current, which was originally established for antennas in free space. The kernel of the integral equation is, however, approximated, and the validity of the solution is limited to restricted ranges of

certain physical and geometric parameters, such as the ratios of the moduli of the propagation constants in the medium and in the dielectric jacket and the ratio of the conductor's to the jacket's diameters, Refs. [4] and [5]. For the end-grounded antennas no solution seems available. The simple idea of considering the antenna as a section of a short-circuited transmission line, Ref. [5] does not take into account the geometry and size of the antenna regions in contact with the medium. Consequently, this simple approach is not useful for the end-grounded, electrically short antenna for which the electrode parameters crucially affect the antenna resistance (see Section 4).

The method used here is not restricted to a limited range of parameters, but applies, with the appropriate modifications, to both the insulated and end-grounded antennas. An expression for the impedance is established which depends upon the Fourier Transforms of the various quantities with respect to the axis of symmetry of the problem. The approach drastically simplifies the computations, transforming into single integrals multiple integrals of convolutional type, which are difficult to evaluate because of the singularities of the integrands. For the insulated case, the current is determined by enforcing the appropriate boundary conditions for the electric field on the antenna conducting surface, via an application of the moment method, Ref. [6], in the wavenumber domain rather than in the coordinate domain (Section 3). The simplification thus obtained is substantial. For the end-grounded antenna, a parallel approach could have been used. However, in this case, for electrically short dipoles, a further simplification of the analysis is permissible (Section 4), due to the possibility of making a reasonably accurate guess of the functional form of the antenna current and to the stationary character of the expression for the antenna impedance.

The analysis is, in principle, limited to antennas in an unbounded medium. Practically, it can be applied to submerged antennas in sea water, provided the distance from the ocean surface to the antenna is greater than a skin depth. In this case, the presence of the surface, as physically clear, has no significant effect on the antenna impedance and current distribution, (Ref. [7]).*

*To understand this, the following heuristic reasoning will help. The antenna impedance in the presence of the water/air interface is equal to that of the antenna in an unbounded ocean plus a correction term due to the energy scattered from the ocean surface. If the radiation of the antenna is represented via a plane wave expansion, each plane wave experiences, after a reflection at the interface, an attenuation - two ways - greater than 17 dB for every skin depth. This corresponds to an error in the impedance calculation of less than $8 \cdot 10^{-2}$, $2 \cdot 10^{-2}$, $3.10^{-3}$ at one, two, and three skin depths, respectively. To put the issue in perspective, the skin depth at 18 Khz is approximately 1.8 m, and depths greater than, say, 5 or 6 m minimum are those of greatest practical interest in this study.

# 2. CURRENT AND IMPEDANCE CALCULATIONS

## 2.1 GENERAL PHILOSOPHY

Although already mentioned in the Introduction, for the sake of clarity it is here restated that the objective of the analysis and the associated computer codes is to find the antenna impedance and the current distribution, in turn related to the thermal noise and sensitivity.

Denote by "e" the antenna radius of the feed point of the antenna, the latter being a small region, a cylindrical gap, interrupting the continuity of the antenna conductor. Because this region is small, the axial component of the electric field is constant in the feed gap. If $\rho$, $z$, and $\phi$ are cylindrical coordinates, the z-directed field in the gap is equal to

$$E_z(\rho=e,\ z,\phi) = \frac{V_o}{\delta}\ \mathrm{rect}\left(\frac{z2}{\delta}\right), \qquad (5)$$

(Fig. 7), where rect(x) is a function equal to 1 for $|x| \leq 1$ and zero for $|x| > 1$. Consider now a structure equal to that of the actual antenna but with the gap eliminated; that is, with the metallic continuity of the antenna conductor reconstituted. Consider the gap field, (5), as a ribbon of magnetic current radiating in the presence of the antenna conductor (with the gap eliminated). The magnetic current must be imagined to be located at $\rho = e^+$, that is, separated, but very close to the antenna cylindrical surface. It can be easily shown on the basis of one of the forms of the equivalence theorem, that the field outside the antenna is not changed in the new situation (depicted in Fig. 7), a fact perhaps intuitive (Appendix B). The problem to

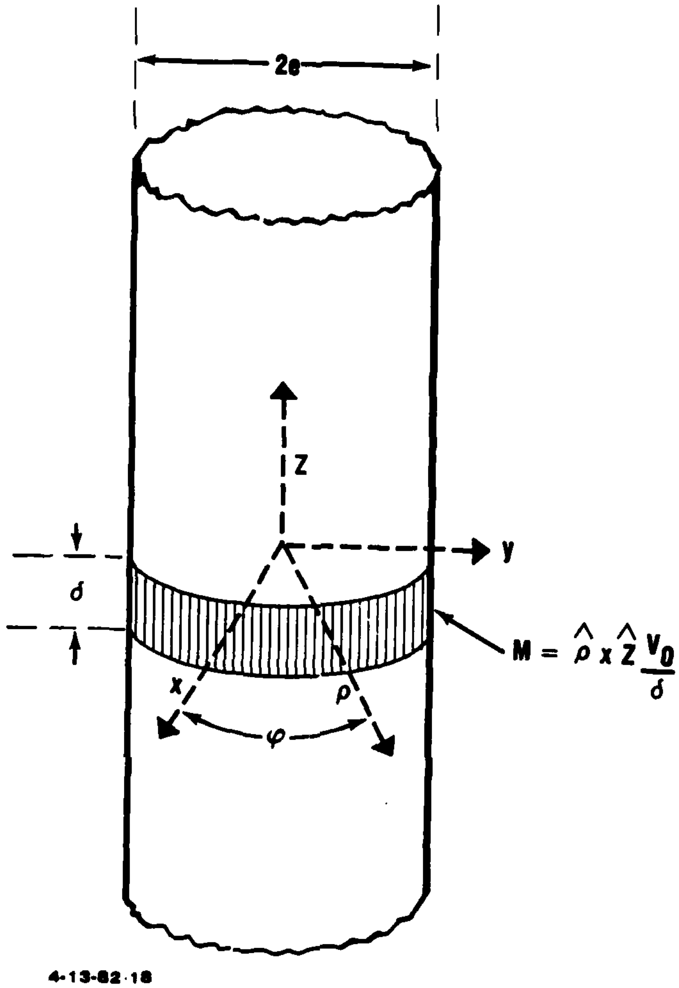


FIGURE 7. Antenna excitation

be solved can be viewed now as a near-field scattering problem: determine the current on the antenna (uninterrupted) conductor when the ribbon of magnetic current

$$\underline{\underline{M}} = \hat{\rho} x \hat{z} \frac{V_o}{\delta} \operatorname{rect}\left(\frac{2z}{\delta}\right) \delta(\rho - e) = \frac{V_o}{\delta} \operatorname{rect}\left(\frac{2z}{\delta}\right) \delta(\rho - e)\, \hat{\phi} \tag{6}$$

is present at $\rho = e^+$. The field (5) is the incident field of the scattering problem, because the ribbon of magnetic current is infinitely close to the antenna surface. The current (to be found) will be the correct one if it supports an electric tangential (z-directed) field equal to zero on the antenna surface except for $|z| < \delta$, where it must be equal in magnitude and opposite in sign to the gap field (5). In computing the field, the current must be assumed, according to the equivalence theorem, as radiating without the conductor present, that is, in homogeneous medium for the case of Figs. 1 and 4, and in a cylindrically layered region in the other cases.

## 2.2 A VARIATIONAL EXPRESSION FOR THE ANTENNA IMPEDANCE

Call $\underline{J}$ the density of current on the surface of the conductor. Let us denote by $L(\underline{J})$ the integro-differential operator - which does not need to be specified for the time being - which, when acting on $\underline{J}$, generates the tangential electric field $\underline{E}_t$ on the conductor surface

$$\underline{E}_t = L(\underline{J}) \quad . \tag{7}$$

The correct current distribution generates a tangential field on the conductor equal in magnitude and opposite in sign to the incident field due to the ribbon of magnetic current. Thus, from (5) and (7), on the antenna surface

$$-\hat{z} \frac{V_o}{\delta} \operatorname{rect}\left(\frac{2z}{\delta}\right) = L(\underline{J}) \ . \tag{8}$$

For obvious reasons of symmetry, the current density is $\hat{z}$ directed and independent of $\phi$

$$\underline{J}\,(z,\phi) = \frac{1}{2\pi e}\, I(z)\hat{z} \quad . \tag{9}$$

A variational expression for the antenna impedance, which is stationary with respect to small variation of $\underline{J}$ with respect to the correct functional form, can be established via the following procedure. Introduce the inner product in a function space

$$\iint_{\sigma} \underline{E}_t \cdot \underline{J}\, d\sigma \equiv \underline{J} \cdot L(\underline{J}) \quad , \tag{10}$$

where $\sigma$ is the surface of the antenna conductor. From (9), one obtains

$$- \frac{V_o}{\delta} \int_{o}^{2\pi} \int_{-\delta/2}^{\delta/2} \frac{I(z)}{2\pi e}\, dz d\phi = \underline{J}\cdot L(\underline{J}) \quad . \tag{11}$$

Because $\delta << h < \lambda$, in (11) $I(z) \approx I(0) \equiv I_o$, yielding

$$- V_o\, I_o = \underline{J}\cdot L(\underline{J}) \quad . \tag{12}$$

Since the antenna impedance is

$$Z = \frac{V_o}{I_o} \; , \tag{13}$$

from (12), it follows that

$$Z = - \frac{\underline{J}\cdot L(\underline{J})}{I_o^2} \quad . \tag{14}$$

Let us take the first variation of (14):

$$\delta Z = -\frac{\delta \underline{J}\cdot L(\underline{J}) + \underline{J}\cdot L(\delta \underline{J})}{I_o^2} + 2\,\frac{\underline{J}\cdot L(\underline{J})}{I_o^3}\,\delta I_o \ . \tag{15}$$

Since L is a symmetric operator (because of reciprocity),

$$\delta \underline{J}\cdot L(\underline{J}) = \underline{J}\cdot L(\delta \underline{J}) \ . \tag{16}$$

Also, from (7) and (8),

$$-V_o\,\delta I_o = \delta \underline{J}\cdot L(\underline{J}) \ . \tag{17}$$

Therefore,

$$\delta Z = \frac{+\,2V_o\,\delta I_o}{I_o^2} + \frac{2\underline{J}\cdot L(\underline{J})}{I_o^3}\,\delta I_o \ . \tag{18}$$

If the current distribution has been varied from the correct one, from (13) and (14) it follows that

$$\delta Z = 0 \ , \tag{19}$$

which shows that the expression (14) for the impedance is insensitive to a "small" (first-order) variation of the current distribution from the correct one. The practical meaning of this result is that the expression (14) of the antenna input impedance is rather forgiving with respect to a somewhat inaccurate guess of the antenna current. Notice that, because of the normalizing term in the denominator of (13), only the functional form of the current is relevant, and not its absolute level.

For the antenna of Fig. 6, in the range of frequencies and physical length of interest for which

$$\frac{h}{\lambda_o} \ll 1 \tag{20}$$

(where $\lambda_o$ is the free space wavelength), the assumption of constant current on the antenna is completely adequate--most of the antenna resistance being of ohmic nature and localized at the electrode/seawater contact. In other cases, for which a guess of the form of the current may not be easy, or when a greater accuracy is sought, the moment method can be applied to determine the current distribution and the antenna impedance.

## 2.3 INTERVENTION OF THE MOMENT METHOD

In the remaining part of this section, a sketchy exposition of the moment method will be given. Again, the essence of the method can be explained in rather abstract terms without the need to discuss the explicit form of the operator $L(\underline{J})$, the latter issue being the subject of the next sections. The moment method can be discussed without reference to the variational expression (14) for the impedance. However, the use of (14) as a starting point provides perhaps a better insight into the nature of the method and the reason for its accuracy in predicting the antenna impedance, which is related to the stationary character of (14).

Postulate that the antenna current distribution $\underline{J}$ is approximated well by a weighted superposition of a finite, and small, number of suitably selected known basis functions $\underline{J}_n$:

$$\underline{J} \simeq \sum_{n=1}^{N} c_n \underline{J}_n \quad . \tag{21}$$

For the structure of interest, a linear antenna, from (9)

$$\underline{J}_n = \frac{1}{2\pi e} \hat{z} \; I_n(z) \quad (n = 1, ..N) \quad . \tag{22}$$

The set $\{I_n(z)\}$ can be chosen suitably as consisting of functions different from zero only in $|z| < h$, see Section 3. Then the expression (14) is written as follows, because of the linearity of L( ):

$$Z = - \frac{\sum_{s=1}^{N} \sum_{k=1}^{N} c_s c_k \; \underline{J}_s \cdot L(\underline{J}_k)}{\left[\sum_{n=1}^{N} c_n I_n(0)\right]^2} \tag{23}$$

By invoking the variational character of (14) it is argued that, if (21) holds, the coefficients $c_n$ must be such as to make (23) stationary. Thus, the derivatives of (23) with respect to each of the coefficients must be equal to zero. This leads, with a little algebra, to the set of N equations:

$$\sum_{k=1}^{N} c_k \; \underline{J}_s \cdot L(\underline{J}_k) = \frac{\left[\sum_{s=1}^{N} \sum_{k=1}^{N} c_s c_k \; \underline{J}_s \cdot L(\underline{J}_k)\right]}{\sum_{n=1}^{N} c_n I_n(0)} I_s(0) \quad (s = 1,..N), \tag{24}$$

but the term in square brackets is equal to $\underline{J} \; L(\underline{J})$, and because of (21) the denominator is equal to I(0). Therefore, for the

correct set of coefficients, from (14) one obtains the N relationships:

$$\sum_{k=1}^{N} c_k \, \underline{J}_s \cdot L(\underline{J}_k) = - V_o \, I_s \,(0) \quad . \tag{25}$$

Consequently, if the notation is introduced,

$$m_{sk} = \underline{J}_s \cdot L(\underline{J}_k) \;, \tag{26}$$

the set of equations (24) is written:

$$\sum_{k=1}^{N} m_{sk} \, c_k = - V_o \, I_s(0) \quad (s = 1, ..N) \,, \tag{27}$$

which gives the set of coefficients $c_k$. Then from (27), (24), and (14)

$$Z = \frac{- V_o}{\sum_{s=1}^{N} c_s I_s(0)} \quad . \tag{28}$$

Since the basis functions $I_n(z)$ can be normalized arbitrarily, their value at $z = 0$ will be chosen equal to unity. The matrix $\underline{\underline{M}}$ is conveniently introduced

$$\underline{\underline{M}} \equiv \left\{ m_{ik} \right\} \quad (i,k = 1, ..N) \;, \tag{29}$$

and $V_o$ is chosen equal to unity. The numerical column vector $\underline{e}$ of dimension N is defined as a string of ones

$$\underline{e} \equiv [1, 1, 1, ..1]^T \,, \tag{30}$$

where T means transpose. The coefficients $c_i$ of the expansion of the current are also considered the components of a numerical vector $\underline{c}$. With this notation and choice of normalization, the fundamental equation (27) becomes

$$\underline{\underline{M}}\,\underline{c} = -\,\underline{e}\ , \qquad (31)$$

that is,

$$\underline{c} = -\,\underline{\underline{M}}^{-1}\cdot\underline{e}\ . \qquad (32)$$

The elements of $\underline{\underline{M}}$ take a relatively simple form if a convenient mathematical representation for the operation L( ) is used. By invoking the azimuthal symmetry of the structure, a diagonalization of L( ) is achieved by representing the antenna field as a superposition of cylindrical waves, each identified by its wavenumber in direction z. As will be apparent, the simplification thus obtained is substantial for an antenna in a homogeneous medium, and becomes drastic and essential for more complex situations, such as those depicted in Figs. 2 and 6. The instances in which the functional form of the antenna current distribution can be guessed *a priori*, can be considered particularly simple and trivial applications of the particular procedure outlined here. In fact, in such cases N = 1, and from (32) and (28) one reobtains (14), as it must be.

## 3. THE BARE AND THE INSULATED ANTENNA IN A DISSIPATIVE MEDIUM. ANALYSIS VIA MOMENT METHOD IN THE WAVENUMBER DOMAIN.

In this section, the analytical structure of the linear operator L( ), symbolically introduced in the previous sections, will be discussed in detail. The objective, of course, is to establish a practical procedure for the numerical calculation of the field supported by a current on the antenna conductor, a necessary ingredient for the calculation of the antenna impedance, according to the recipe expressed by (14) or (28).

The antenna metal part is modeled as a conducting tube with a thin wall. The assumption that the antenna is tubular rather than solid makes the mathematics neater and simpler. On the other hand, the physics of the phenomena clearly suggests that in any practical sense the current and impedance calculated for the tubular model are not different from those pertaining to the case of a solid conducting rod, provided $2e \ll h < \lambda$. For the insulated antenna only, and not for the antenna with electrodes depicted in Fig. 6, another simplifying assumption is introduced: the dielectric jacket in the idealized model is considered to extend axially well beyond the limits of the antenna conductor (Fig. 8). Since the current goes to zero at the antenna end, it is clear that the difference in the antenna impedance and the current distribution is negligible in the two cases. (It would not be so for a very thick rod.) With this assumption, all the cases in Figs. 1 to 5--dipole in a dielectric medium (free space), dipole in a conducting medium, and dipole coated by an insulating jacket in a dissipative medium--can be lumped together in the analysis and treated as particular cases of a more general class of structures, defined as follows.

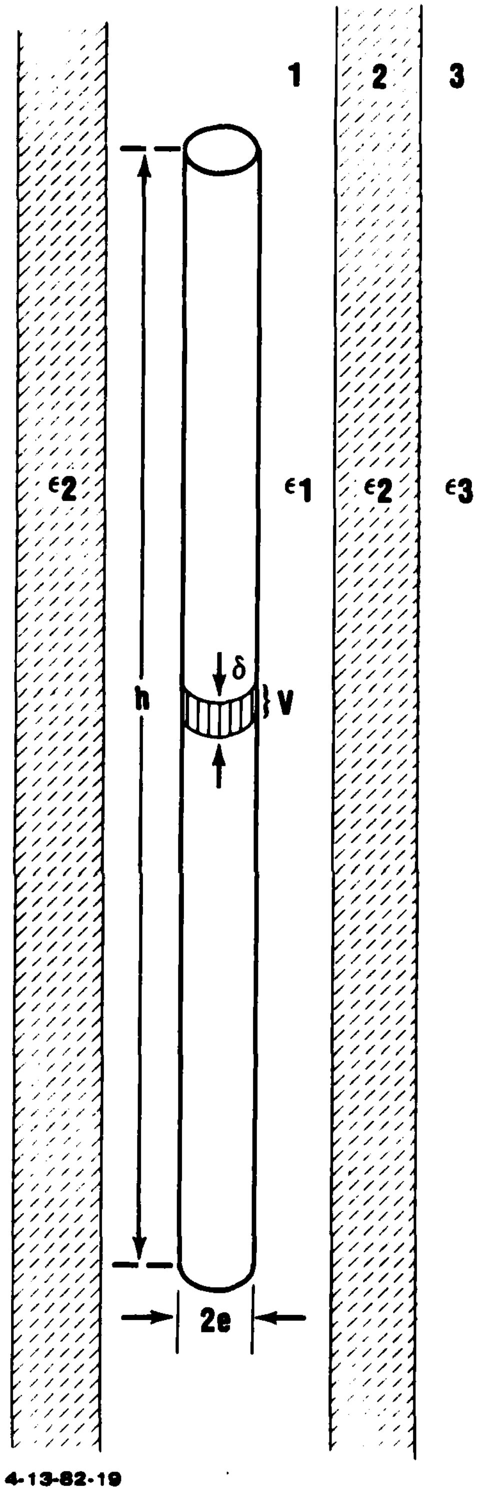


FIGURE 8. Idealized physical model for the insulated antenna

A tubular conductor with thin walls excited by a ribbon of magnetic current at the gap (see Section 2.1), is immersed in a radially inhomogeneous medium whose dielectric constant is, in general, a complex quantity, to accommodate losses, and is a function of the radius:

$$\varepsilon(\rho) = \varepsilon_r(\rho) + j\varepsilon_j(\rho) \ , \tag{33}$$

with

$$\varepsilon_j(\rho) = -\frac{\sigma_j(\rho)}{\omega\varepsilon_0} \ , \tag{34}$$

where $\varepsilon_0$ is the permittivity of free space. In practical cases (e.g., an antenna in water with an insulating jacket), $\varepsilon(\rho)$ will be a stepwise function, with very few steps, as in Fig. 8.

Clearly, because of the azimuthal symmetry, only the z and $\rho$ components of the electric field and the $\phi$ component of the magnetic field will be different from zero. For $\rho \neq e$, Maxwell's equation yields then the three relationships

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = -j\omega\mu H_\phi \ , \tag{35}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi) = j\omega\varepsilon E_z \ , \tag{36}$$

and

$$-\frac{\partial H_\phi}{\partial z} = j\omega\varepsilon E_\rho \ , \tag{37}$$

where $\varepsilon$ is a function of $\rho$, and the field components are functions of $\rho$ and z. Equations (35), (36), and (37) hold in both

the regions $\rho > e$ and $\rho < e$. At $\rho = e$, the magnetic field has a jump equal to the surface current density

$$H_\phi(z,e^+) - H_\phi(z,e^-) = \frac{I(z)}{2\pi e} , \qquad (38)$$

where $e^+$ and $e^-$ are values of $\rho$ greater or smaller than, but infinitesimally close to e. It is recalled that I(z) in (38) is a function different from zero only for $|z| \leq h/2$.

The first step for the solution of the problem at hand is the introduction of the Fourier Transforms (FTs) with respect to z of the various quantities, denoted here by the same letter as the quantities themselves, with however, the addition of a caret "^". For example,

$$\hat{E}_z(w,\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E_z(z,\rho)\ e^{jwz}dz \qquad (39)$$

and parallel definitions hold for $\hat{E}_\rho(w,\rho)$, $\hat{H}_\phi(w,\rho)$, and $\hat{I}(w)$. The inversion of (39) gives:

$$E_z(z,\rho) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{E}_z(w,\rho)\ e^{-jwz}dw , \qquad (40)$$

and again, parallel expressions hold for the other quantities. According to (40) and its companions, the electromagnetic field is represented as a superposition of azimuthally symmetric cylindrical waves, each identified by a wavenumber w in the z direction.

In Appendix C, it is shown that Eqs. (35) and (36) become, by using the FTs of the various functions of z

$$\frac{d\hat{E}_z}{d\rho} = j\omega\mu \frac{k^2-w^2}{k^2} \hat{H}_\phi \tag{41}$$

and

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho\hat{H}_\phi) = j\omega\varepsilon\hat{E}_z \quad , \tag{42}$$

where the FTs of the field components are clearly functions of w and $\rho$, $\hat{I}$ is a function of the longitudinal wavenumber w, and $\varepsilon$ and

$$k = \omega\sqrt{\mu\varepsilon} \tag{43}$$

are functions of $\rho$. In (43), the branch of the root must be chosen in such a way that

$$\mathrm{Re}[k] > 0 \quad , \tag{44}$$

consistent with the assumed time dependence exp $j\omega t$. In deriving (41), the FT of (37),

$$\frac{w}{\omega\varepsilon}\hat{H}_\phi = \hat{E}_\rho \quad , \tag{45}$$

has been used.

For each value of w, and each particular stratification profile $\varepsilon(\rho)$, (41) and (42) constitute a system of ordinary differential equations whose solution for $\hat{H}_\phi$ has a jump at $\rho$ = e equal to the FT of the current. For the simple case of a

homogeneous medium, the solution is well known (see example below). In general, for each $\rho \neq e$, a wave impedance can be introduced

$$\frac{\hat{E}_z(w,\rho)}{\hat{H}_\phi(w,\rho)} = z(w,\rho) \tag{46}$$

which does not depend, of course, upon the source $\hat{I}(w)$, being associated with the source-free solution of the system (41) and (42). The FT of (38), with self-explanatory notation is

$$\hat{H}_\phi^{\,+} - \hat{H}_\phi^{\,-} = \frac{\hat{I}}{2\pi e} \quad , \tag{47}$$

where the arguments of the functions have been dropped.

According to the discussion in Section 2.1, the current $I(z)$ must support a tangential field $\hat{E}_z(z,\rho)$ equal to zero on the antenna conducting surface, except in the gap region, where the tangential electric field must be equal in magnitude and opposite in sign to the impressed field, see (8).

To proceed, notice that $\hat{E}_z(w,\rho)$ is continuous at $\rho = e$. Thus, the wave impedances at the conductor surface, on its exterior and interior sides, are, from (46)

$$z^+(w) \equiv z(w,e^+) = \frac{\hat{E}_z(w,e)}{\hat{H}_\phi(w,e^+)} \tag{48}$$

and

$$z^-(w) \equiv z(w,e^-) = \frac{\hat{E}_z(w,e)}{\hat{H}_\phi(w,e^-)} \quad , \tag{49}$$

respectively. Then from (47) we obtain:

$$\hat{E}_z(w,e) = \frac{\hat{I}(w)}{2\pi e} \frac{z^+z^-}{z^- - z^+} . \tag{50}$$

[If so inclined, one can interpret (50) as the voltage generated by a current generator of strength $-\hat{I}(w)$ feeding two parallel circuits having impedance $-z^+$ and $z^-$, not an exceedingly useful exercise.] The important point is that analytical expressions for $z^+(w)$ and $z^-(w)$ can be established. The difficulty of their evaluation ranges from almost nil (for homogeneous medium) to moderate for several thin layers. The general discussion of this issue is, however, beyond the scope of this report.

We pause now, for the benefit of the reader who wants to understand the method by thinking in less abstract terms, to recall the analytical expression of $z(w,\rho)$ for the simplest case, an antenna in a homogeneous, possibly dissipative medium.

Example 1. Bare Antenna

The solution of (41) and (42) for a homogeneous medium, that is, for $\varepsilon$ and $k$ independent of $\rho$, is found in any textbook on electromagnetics. It is recalled that the integration is obtained by eliminating $\hat{H}_\phi$ between (41) and (42), solving the standard wave equation so obtained, and then using (41) to determine $\hat{H}_\phi$ (see for example, pages 198-216 of Ref. [8]). The solution has, of course, different analytical form for $\rho > e$ and $\rho < e$, because of the need to satisfy in the two regions the radiation conditions and the regularity at the origin $\rho = 0$, respectively. In the two regions the ratio (46) is found to be

$$z(w,\rho) = \frac{-j\sqrt{k^2-w^2}}{\omega\varepsilon} \frac{H_0^{(2)}(\rho\sqrt{k^2-w^2})}{H_1^{(2)}(\rho\sqrt{k^2-w^2})} \tag{51}$$

for $\rho > e$, and

$$z(w,\rho) = \frac{-j\sqrt{k^2-w^2}}{\omega\varepsilon} \frac{J_o(\rho\sqrt{k^2-w^2})}{J_1(\rho\sqrt{k^2-w^2})} \tag{52}$$

for $\rho < e$. Of course, $z^+(w)$ and $z^-(w)$ are obtained from (51) and (52) by setting $\rho = e$. In order to satisfy radiation conditions, the branch of the square root in the argument of the cylindrical functions must be chosen as follows:

$$\mathrm{Re}\left[\sqrt{k^2-w^2}\right] > 0 \quad . \tag{53}$$

Because of (44), (53) guarantees that

$$-\frac{\pi}{2} < \mathrm{Arg}\sqrt{k^2-w^2} < 0 \quad . \tag{54}$$

In connection with (51) and (52), a few observations are in order:

For slender antennas (that is, for $e << h$), it is

$$|z^+(w)| << |z^-(w)| \tag{55}$$

for w in the range of interest which is, roughly, $|w|$ less than several times the inverse of the antenna length; in fact, the behavior of $z^+(w)$ and $z^-(w)$ for larger wavenumbers is unimportant because $\hat{I}(w)$ becomes comparatively negligible, being Fourier Transform of $I(z)$ which is a smooth function different from zero only for $|z| < h/2$. Therefore, (50) becomes

$$\frac{z^+z^-}{z^--z^+} \simeq z^+ \quad , \tag{56}$$

an approximation that, for solid rod antennas, perhaps not surprisingly, gives better fit to experimental data on current distribution and impedance.

On the other hand, by using the Wronskian relationship for cylindrical functions,

$$J_1(z)Y_o(z) - J_o(z)Y_1(z) = \frac{2}{\pi z} , \qquad (57)$$

it can be promptly shown that in the case here addressed of a dipole in a homogeneous medium

$$\frac{1}{2\pi e} \frac{z^+z^-}{z^- - z^+} = - \frac{k^2-w^2}{4\omega\varepsilon} J_o(e\sqrt{k^2-w^2})\, H_o^{(2)}(e\sqrt{k^2-w^2}) ,$$

an equation that can be reestablished via a completely different approach as is done in Appendix D. This computation of the cylindrical functions for complex argument does not create any numerical problem. In fact, for the modulus of the argument less than approximately 20, a series expansion is used, whereas, for greater values of the argument, a standard asympotic expansion is available. This ends the discussion of this example.

---

We are now in the position to write (8) more explicitly. To simplify the notation, the function is defined

$$z(w) \equiv \frac{1}{2\pi e} \frac{z^+(w)\; z^-(w)}{z^-(w) - z^+(w)} . \qquad (59)$$

Introduce (59) into (50) and take its inverse Fourier Transform. One finds that for $|z| < \delta/2$,

$$-\frac{V_o}{\delta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{I}(w)\; z(w) e^{-jwz} dw , \qquad (60a)$$

and for $\delta/2 \leq |z| \leq h/2$,

$$0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{I}(w)\ z(w) e^{-jwz} dw. \qquad (60b)$$

By using the symbol rect ( ), (60a) and (60b) can be written compactly as follows:

$$-\frac{V_o}{\delta} \text{rect} \frac{2z}{\delta} = \text{rect} \frac{2z}{h} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{I}(w)\ z(w) e^{-jwz} dw. \qquad (60)$$

If we postulate that, not only in homogeneous media, but in more general situations the simplifications (56) hold in the range of wavenumbers of interest (see previous example), then (59) becomes

$$z(w) \approx \frac{1}{2\pi e} z^{+}(w) \quad . \qquad (59a)$$

The right side of (60) defines the nature of the operator L( ), introduced symbolically in Section 2.1, if the explicit expression of z(w) is shown.

Equation (60) is an integral equation for the FT of the antenna current. Notice that (60) holds for $-h/2 < z < h/2$ only: this crucial fact makes its solution non-trivial and not given simply by an inverse FT.* Notice also that the information on the nature of the medium surrounding the antenna is totally summarized in the particular z(w) germaine to the specific problem under consideration. The case of a homogeneous

*The situation is analogous to that in the well-known Weiner-Hopf equation for which the range of the variable is zero to infinity.

medium has been considered in Example 1 and the expression of $z(w)$ for the more complicated situation of the antenna with an insulating jacket in a dissipative medium will be given in Example 2 at the end of this section, more general configurations being briefly discussed in Appendix E.

We are now in the position to apply the moment method, introduced in Section 2.3 for an operator L( ) unspecified, to the solution of (60).

Write the current as a superposition of components $I_n(z)$, where $I_n(z)$ are functions different from zero only for $|z| < h/2$, see (21) and (22):

$$I(z) = \sum_{n=1}^{N} c_n I_n(z) \quad . \tag{61}$$

Since the current goes to zero at the antenna ends, that is, at $z = \pm h/2$, a convenient set of basis functions is

$$I_n(z) = \text{rect}\left(\frac{2z}{h}\right) \cos\left(\frac{2n-1}{h}\pi z\right) \quad , \tag{62}$$

whose FTs are easily found to be

$$I_n(w) = \frac{\pi h}{2} \frac{(-1)^n}{\sqrt{2\pi}} \frac{(2n-1)\cos\frac{wh}{2}}{\left(\frac{wh}{2}\right)^2 - \left[(2n-1)\frac{\pi}{2}\right]^2} \quad . \tag{63}$$

The elements of the matrix $\underline{\underline{M}}$ are obtained according to the prescription (26), with L( ) defined by the right side of (60). The result, as shown in Appendix F, is

$$m_{sk} = \int_{-\infty}^{+\infty} \hat{I}_s(-w)\; z(w)\; \hat{I}_k(w) dw \quad . \tag{64}$$

The coefficients of the current expansion are then given by (32).

Prior to discussing in Example 2 a practical case, it is worth pausing to examine the analytical and computational nature of the solution. The elements of $\underline{\underline{M}}$ involve simple integrals extended to a domain, theoretically infinite, but practically limited to the region where all $\hat{I}_n(w)$'s are not negligible with respect to their maximum values. The key idea, which makes the approach analytically and computationally simple, is the evaluation of the elements of $\underline{\underline{M}}$ via integrals in the wavenumber domain rather than in the coordinate domain. Had the computations of the elements of $\underline{\underline{M}}$ been done in the coordinate domain using directly (26), multiple integrals of convolutional type should have been evaluated. Besides the greater numerical complexity, those integrals require careful analytical manipulations before attempting their numerical evaluations, because of the presence of singularities in their integrands. None of these problems exists in the method here adopted.

**Example 2. Insulated Antenna**

The structure considered is that shown in Figs. 3 and 4. Let $\varepsilon_1$ and $\varepsilon_2$ be the dielectric constant, generally complex, of the insulating jacket and of the surrounding medium. Denote by 2p the diameter of the insulating jacket. As is practically always the case, the assumption will be made that

$$|k_1 p| << 1 \quad ; \quad |k_2 p| << 1 \quad , \tag{65}$$

which allows one to establish a remarkably simple approximate expression for $z(w,e^+)$. In fact, if (65) holds, inside the dielectric jacket with extremely good approximation

$$H_\phi(z,\rho) \simeq \frac{I(z)}{2\pi\rho} \quad , \tag{66}$$

and consequently

$$\hat{H}_\phi(w,\rho) \simeq \frac{\hat{I}(w)}{2\pi\rho} \quad , \tag{67}$$

which, inserted in (41), with $k = k_1$, yields the law of variation of $E_z$ inside the jacket

$$d\hat{E}_z = j\omega\mu \frac{k_1^2 - w^2}{k_1^2} \frac{\hat{I}(w)}{2\pi} \frac{d\rho}{\rho} . \tag{68}$$

By integrating from $\rho = e$ to $\rho = p$

$$\hat{E}_z(w,e) = \hat{E}_z(w,p) - \frac{j\omega\mu}{2\pi} \hat{I}(w) \log\left(\frac{p}{e}\right) \frac{k_1^2 - w^2}{k_1^2} . \tag{69}$$

Divide both sides of (69) by $\hat{H}_\phi(w,e)$. Recall that (67) implies that inside the insulating jacket

$$\hat{H}_\phi(w,e) = \hat{H}_\phi(w,p) \frac{p}{e} . \tag{70}$$

Then, from (69) and the definition (48)

$$z^+(w) = \frac{e}{p} z(w,p) - j\omega\mu e \log \frac{p}{e} \frac{k_1^2 - w^2}{k_1^2} . \tag{71}$$

where $z(w,p)$ is the wave impedance at $\rho = p$ for an outgoing wave in the dissipative medium, whose expression is given by (51) if we put $\rho = p$ and $k = k_2$. Equation (71) is then used in the expressions (64) of the elements of the matrix $\underline{M}$.

---

With the discussion of Example 2, all the ingredients are now available for the evaluation of the current and impedance of insulated antennas in a conducting medium.

To end this section, a few remarks about the validation of the formalism and the concomitant computer code for the analysis of the insulated antenna are in order. Of course, it would be

highly desirable to achieve a complete validation by comparing computed results and measurements for several configurations. This task appears to be difficult, however, because scarce experimental data on current and impedance of insulated antennas in dissipative media exist, but they are presented in a way that makes it difficult to understand what the pertaining physical and geometrical parameters are (Refs. [3], [4], and [5]). One has to be content with a partial validation based on the argument that the perfect current and impedance prediction for the case of antenna in free space, see below, is a good indication of the validity of the approach for more complicated situations. The reason lies, as discussed above, in the capability of the formalism to handle different structures belonging to the class shown in Fig. 8 in a unified way, the only difference being the definition of the functional form of z(w) in (64).

In Fig. 9, the current and impedance for a dipole in free space are shown. The experimental data for the same set of geometrical and physical parameters are shown in the insert in the upper right corner of the figure, as is the current distribution computed with King's three-term theory. The perfect agreement with the experimental data is clearly apparent.

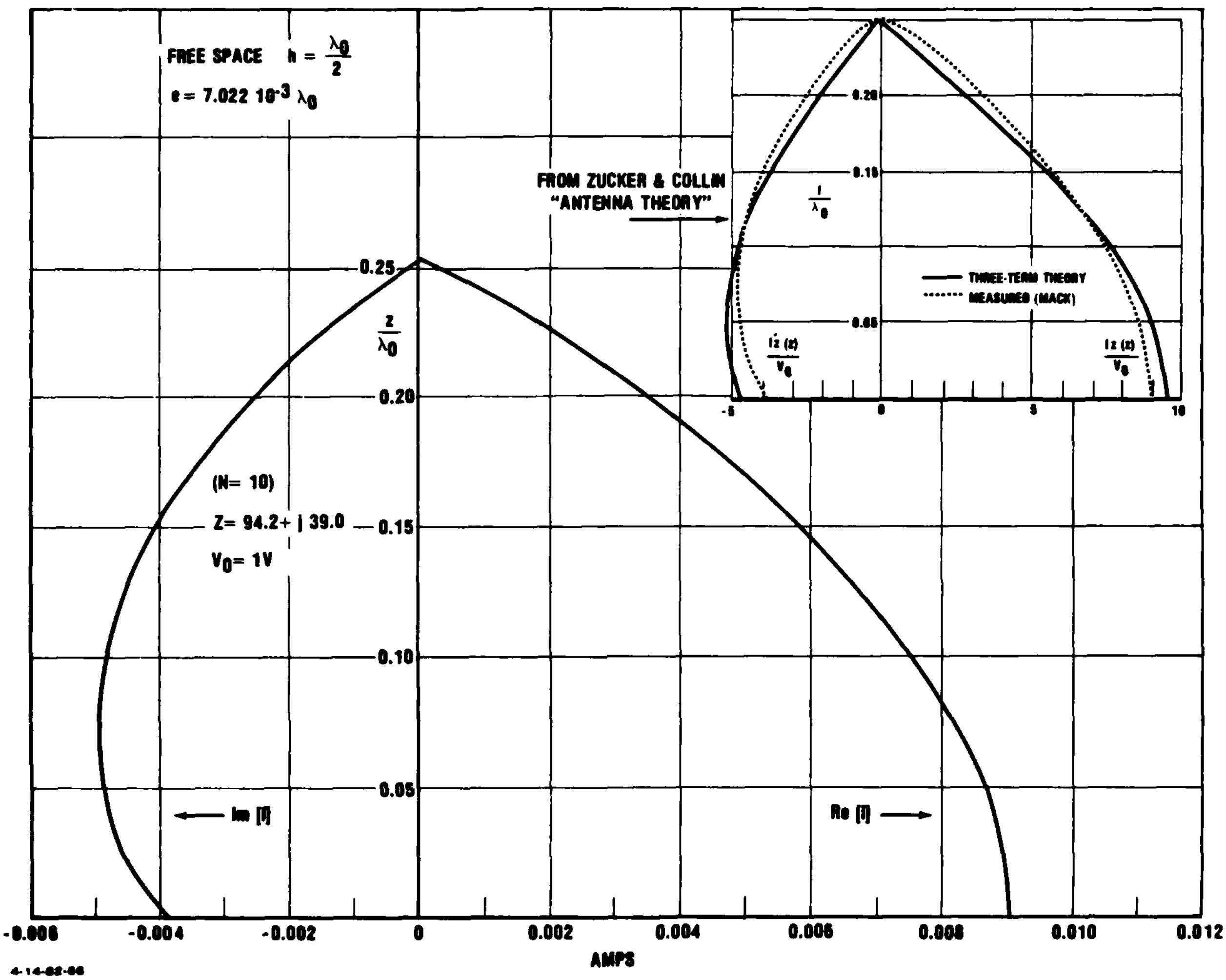


FIGURE 9. Calculated current and impedance for an antenna near resonance in free space

## 4. END-GROUNDED ANTENNA IN A DISSIPATIVE MEDIUM

### 4.1 ANALYSIS

The configuration studied here is depicted in Fig. 6. The antenna consists of a cylindrical segment of a cable ending with two terminal sections in contact with the water. Unlike the insulated antenna considered previously in this configuration, the antenna is assumed to be fed at one end (as it would be in any practical application). The electrode's diameter is equal to the diameter "2p" of the cable. The electrode's length is denoted by "L" - a parameter of crucial importance in determining the antenna impedance if the overall antenna length is "small", that is, if $h \ll \lambda_o$.

In the case of interest, for which $|\varepsilon_2| \gg |\varepsilon_1|$, a guess of the form of the current is easy to make. In fact, the antenna structure can be heuristically likened to a section of a lossy coaxial line, short circuited at the end, the dissipative medium constituting its outer conductor. Clearly, for very short antenna lengths a current constant on the antenna insulated part and linearly going to zero along the electrode's length is an excellent approximation*. A slightly better guess

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*The heuristic justification of the assumption of a linear current decay is the following. The electrode is short with respect to skin depth in the medium. Therefore, the current decay on the electrodes is approximately the same as for the dc (static) case. For the latter, the equipotential surfaces are approximately cylindrical because $L \gg p$. Therefore, the radial electric field on the electrodes and therefore the radial current injected into the dissipative medium must be constant along the electrode length. This fact implies a linear decay of the longitudinal current on the electrode surface (because of the equation of continuity for the current).

for the functional form of the current on the insulated section of the antenna would approximate it by the current on a short-circuited section of the transmission line having the cross section shown in Fig. 10. Under this assumption, if $\gamma$ is the complex propagation constant of the line, the antenna current is immediately found to be

$$I(z) = \frac{\cos[\gamma(h-z)]}{\cos\gamma h} \quad , \tag{72}$$

which, neglecting second-order terms for $|\gamma h| << 1$, is approximated by

$$I(z) \approx 1 \quad . \tag{73}$$

Guessing *a priori* the functional form of the antenna current distribution is equivalent to representing the current with a single term in the current expansion (21), or equivalently, to applying directly (14) (see the end of Section 2.3).

Call the currents on the insulated region of the antenna conductor and on the two electrodes $I_c(z)$, $I_{e1}(z)$, and $I_{e2}(z)$, respectively. The FT of the antenna current is, with self-explanatory notation

$$\hat{I}(w) = \hat{I}_c(w) + \hat{I}_{e1}(w) + \hat{I}_{e2}(w) \quad , \tag{74}$$

the explicit expressions for (74) being given in Appendix G.

Under the excellent approximation (67)

$$\hat{H}_\phi(w,p) = \frac{\hat{I}(w)}{2\pi p} \quad , \tag{75}$$

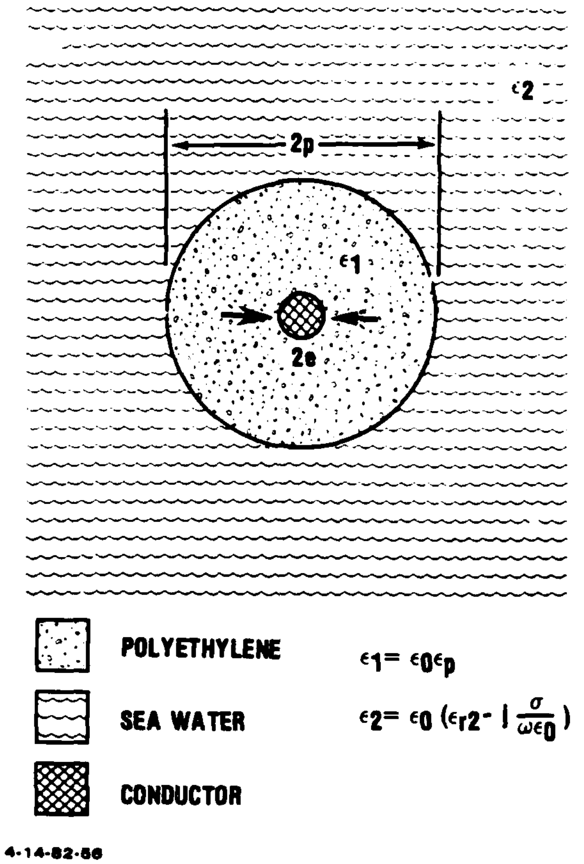


FIGURE 10. Cross section of the cable - sea water guiding structure

which implies

$$\hat{E}_z(w,p) = \frac{\hat{I}(w)}{2\pi p} z(w,p) , \tag{76}$$

with z(w,p), from (51)

$$z(w,p) = \frac{-j\sqrt{k_2^2-w^2}}{\omega\varepsilon_2} \frac{H_o^{(2)}\left(p\sqrt{k_2^2-w^2}\right)}{H_1^{(2)}\left(p\sqrt{k_2^2-w^2}\right)} , \tag{77}$$

where

$$\varepsilon_2 = \varepsilon_o \left(\varepsilon_{2r} - j\frac{\sigma}{\omega\varepsilon_o}\right) \tag{78}$$

and

$$k_2 = \omega\sqrt{\mu\varepsilon_2} \quad . \tag{79}$$

In (79),

$$I_m \left[k_2\right] < 0 \tag{80}$$

and

$$-\frac{\pi}{2} < \arg\sqrt{k_2^2 - w^2} < 0 \ . \tag{81}$$

The application of (14) gives formally for the antenna impedance, recalling that the input current is assumed equal to unity,

$$z = -\int_{\text{electrodes}} \left[I_{e1}(z) + I_{e2}(z)\right] \frac{1}{2\pi p} \ E_z(z,p) \ d\sigma \ +$$

$$-\int_{\text{conductor}} I_c(z) \ \frac{1}{2\pi e} \ E_z(z,e) \ d\sigma \quad , \tag{82}$$

which can be rearranged as the sum of two terms

$$z = z_1 + \Delta z \ , \tag{83}$$

with

$$z_1 = -\int_{-L}^{h+L} \left[ I_{e1}(z) + I_{e2}(z) + I_c(z) \right] dz \; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{E}_z(w,p) e^{-jwz} dw \quad (84)$$

(having introduced the FT of $E_z(z,p)$), and

$$\Delta z = \int_0^h I_c(z) \left[ E_z(z,p) - E_z(z,e) \right] dz \quad . \qquad (85)$$

The evaluation of $z_1$ is now straightforward. By interchanging the orders of integration in (84), one obtains [see (74)]

$$z_1 = -\int_{-\infty}^{+\infty} \hat{E}_z(w,p) \; \hat{I}(-w) \; dw \quad . \qquad (86)$$

Equation (76) provides the relationship between the FT of the current and that of the electric tangential field at $\rho = p$. Consequently, (86) takes the form

$$z_1 = -\frac{1}{2\pi p} \int_{-\infty}^{+\infty} \hat{I}(w) \; z(w,p) \; \hat{I}(-w) \; dw \; . \qquad (87)$$

The evaluation of the "correction term" $\Delta z$ is more complicated and is postponed to Appendix H, where it is shown that $\Delta z$ is equal to

$$\Delta z = \frac{j\omega\mu}{2\pi} \log\left(\frac{p}{e}\right) \int_0^h I_c^2(z) \; dz \; , \qquad (88)$$

an expression that perhaps is intuitively satisfactory, representing an inductive term related to the magnetic energy stored in the end-grounded cable.

The analysis conducted so far assumes that the ohmic resistance of the antenna conductor and electrodes is zero (but takes into account, of course, the ohmic resistance of the contact between electrodes and sea water). However, for antennas with "good" electrodes, that is, having a low contact resistance, the ohmic resistance of the wire, although a relatively small fraction of the total impedance, cannot be totally neglected (see Section 4.2 and 5). An approximate heuristic way of taking it into account is to increase the resistance of the antenna by a quantity

$$\Delta r = \frac{1}{|I(0)|^2} \, r \int_0^h |I(z)|^2 \, dz \quad , \tag{89}$$

where r is the wire resistance per meter. The expression (89) is, in a way, *ad hoc*, and does not fit well in the theory developed so far. However, it has a clear physical interpretation. According to it, the current in the perfectly conducting wire is assumed to be not too different from the current on the actual wire having a finite resistance. This makes (89) a good approximate estimate of the ohmic loss on the wire. Clearly, this is true for good conductors and small antenna lengths, as in the numerical cases discussed here below.

## 4.2 NUMERICAL CASES

In this section selected numerical results are presented for different values of antenna length, electrode length, cable diameter, and frequencies. In Tables 1 to 4, the antenna impedance is shown as a function of the frequency for an antenna having a length of h = 30.5. Also shown are the terms $z_1$, $\Delta z$, and $\Delta r$, whose relative values vs. frequency may be of some interest, being related to different physical phenomena. It is clearly apparent that at low frequency the size of the electrodes, related to their contact resistance, (see also Section 5)

TABLE 1. ANTENNA IMPEDANCE VS. FREQUENCY. ELECTRODE LENGTH = 3.5 cm
(h = 30.5 m, 2p = 16.5 mm, 2e = 1.3 mm, $\varepsilon_p$ = 1.65, r = 0.0134 ohm/m)

| f(kHz) | $z_1$(ohms) Re($z_1$) | I($z_1$) | $\Delta z$(ohms) Re($\Delta z$) | I($\Delta z$) | $\Delta r$ | z(ohm) R | X |
|---|---|---|---|---|---|---|---|
| 10. | 3.248 | 2.067 | .000 | .976 | .406 | 3.654 | 3.042 |
| 20. | 3.422 | 3.887 | .000 | 1.952 | .406 | 3.829 | 5.839 |
| 30. | 3.577 | 5.612 | .000 | 2.930 | .407 | 3.983 | 8.542 |
| 40. | 3.720 | 7.275 | .001 | 3.911 | .407 | 4.128 | 11.186 |
| 50. | 3.851 | 8.900 | .002 | 4.894 | .407 | 4.260 | 13.794 |
| 60. | 3.997 | 10.491 | .003 | 5.880 | .408 | 4.408 | 16.371 |
| 70. | 4.136 | 12.057 | .005 | 6.871 | .409 | 4.549 | 18.928 |
| 80. | 4.283 | 13.603 | .007 | 7.867 | .409 | 4.700 | 21.470 |
| 90. | 4.430 | 15.134 | .010 | 8.868 | .410 | 4.850 | 24.002 |
| 100. | 4.596 | 16.653 | .014 | 9.876 | .411 | 5.021 | 26.528 |
| 110. | 4.770 | 18.163 | .018 | 10.890 | .412 | 5.201 | 29.053 |
| 120. | 4.956 | 19.657 | .024 | 11.912 | .413 | 5.393 | 31.579 |
| 130. | 5.155 | 21.166 | .031 | 12.942 | .414 | 5.601 | 34.108 |
| 140. | 5.373 | 22.664 | .039 | 13.981 | .416 | 5.826 | 36.645 |
| 150. | 5.613 | 24.161 | .048 | 15.030 | .417 | 6.078 | 39.191 |
| 160. | 5.851 | 25.659 | .059 | 16.090 | .419 | 6.329 | 41.749 |
| 200. | 6.953 | 31.691 | .118 | 20.448 | .426 | 7.497 | 52.139 |
| 250. | 8.682 | 39.430 | .241 | 26.227 | .437 | 9.359 | 65.656 |
| 300. | 10.825 | 47.628 | .440 | 32.486 | .451 | 11.717 | 80.114 |

TABLE 2. ANTENNA IMPEDANCE VS. FREQUENCY. ELECTRODE LENGTH = 5 cm
(h = 30.5 m, 2p = 16.5 mm, 2e = 1.3 mm, $\varepsilon_p$ = 1.65, r = 0.0134 ohm/m)

| | $z_1$(ohms) | | z(ohms) | | | z(ohm) | |
|---|---|---|---|---|---|---|---|
| f(kHz) | Re($z_1$) | I($z_1$) | Re(Δz) | I(Δz) | Δr | R | X |
| 10. | 2.775 | 2.067 | .000 | .976 | .406 | 3.181 | 3.043 |
| 20. | 2.963 | 3.888 | .000 | 1.952 | .406 | 3.389 | 5.341 |
| 30. | 3.182 | 5.615 | .000 | 2.930 | .407 | 3.589 | 8.545 |
| 40. | 3.372 | 7.231 | .001 | 3.911 | .407 | 3.780 | 11.191 |
| 50. | 3.559 | 8.905 | .002 | 4.894 | .407 | 3.968 | 13.798 |
| 60. | 3.738 | 10.496 | .003 | 5.880 | .408 | 4.148 | 16.375 |
| 70. | 3.924 | 12.062 | .005 | 6.871 | .409 | 4.338 | 18.933 |
| 80. | 4.112 | 13.609 | .007 | 7.867 | .409 | 4.529 | 21.476 |
| 90. | 4.289 | 15.141 | .010 | 8.868 | .410 | 4.709 | 24.009 |
| 100. | 4.458 | 16.600 | .014 | 9.876 | .411 | 4.913 | 26.536 |
| 110. | 4.680 | 18.171 | .018 | 10.890 | .412 | 5.111 | 29.061 |
| 120. | 4.905 | 19.676 | .024 | 11.912 | .413 | 5.342 | 31.587 |
| 130. | 5.105 | 21.176 | .031 | 12.942 | .414 | 5.551 | 34.118 |
| 140. | 5.346 | 22.674 | .039 | 13.981 | .416 | 5.800 | 36.655 |
| 150. | 5.565 | 24.172 | .048 | 15.030 | .417 | 6.051 | 39.202 |
| 160. | 5.825 | 25.670 | .059 | 16.090 | .419 | 6.302 | 41.760 |
| 200. | 6.956 | 31.705 | .118 | 20.448 | .426 | 7.499 | 52.154 |
| 250. | 8.685 | 37.449 | .241 | 26.227 | .437 | 9.363 | 65.676 |
| 300. | 10.631 | 47.650 | .440 | 32.486 | .451 | 11.722 | 80.137 |

TABLE 3. ANTENNA IMPEDANCE VS. FREQUENCY. ELECTRODE LENGTH = 30 cm
(h = 30.5 m. 2p = 16.5 mm, 2e = 1.3 mm, $\epsilon_p$ = 1.65, r = 0.0134 ohm/m)

| | $z_1$(ohms) | | $\Delta z$(ohms) | | | z(ohm) | |
|---|---|---|---|---|---|---|---|
| f(kHz) | Re($z_1$) | I($z_1$) | Re($\Delta z$) | I($\Delta z$) | $\Delta r$ | R | X |
| 10. | 1.080 | 2.062 | .000 | .976 | .406 | 1.486 | 3.058 |
| 20. | 1.363 | 3.916 | .000 | 1.952 | .406 | 1.770 | 5.666 |
| 30. | 1.646 | 5.654 | .000 | 2.930 | .407 | 2.053 | 8.564 |
| 40. | 1.932 | 7.331 | .001 | 3.911 | .407 | 2.340 | 11.242 |
| 50. | 2.217 | 8.966 | .002 | 4.894 | .407 | 2.626 | 13.859 |
| 60. | 2.502 | 10.568 | .003 | 5.880 | .408 | 2.912 | 16.448 |
| 70. | 2.796 | 12.144 | .005 | 6.871 | .409 | 3.209 | 19.016 |
| 80. | 3.080 | 13.701 | .007 | 7.867 | .409 | 3.496 | 21.568 |
| 90. | 3.377 | 15.243 | .010 | 8.868 | .410 | 3.797 | 24.111 |
| 100. | 3.675 | 16.772 | .014 | 9.876 | .411 | 4.101 | 26.648 |
| 110. | 3.961 | 18.292 | .018 | 10.890 | .412 | 4.391 | 29.162 |
| 120. | 4.264 | 19.806 | .024 | 11.912 | .413 | 4.701 | 31.718 |
| 130. | 4.570 | 21.316 | .031 | 12.942 | .414 | 5.015 | 34.258 |
| 140. | 4.879 | 22.823 | .039 | 13.981 | .416 | 5.333 | 36.805 |
| 150. | 5.192 | 24.330 | .048 | 15.030 | .417 | 5.657 | 39.361 |
| 160. | 5.508 | 25.839 | .059 | 16.090 | .419 | 5.985 | 41.928 |
| 200. | 6.850 | 31.919 | .118 | 20.448 | .426 | 7.393 | 52.367 |
| 250. | 8.700 | 39.727 | .241 | 26.227 | .437 | 9.377 | 65.955 |
| 300. | 10.860 | 47.977 | .440 | 32.486 | .451 | 11.751 | 80.463 |

TABLE 4. ANTENNA IMPEDANCE VS. FREQUENCY. ELECTRODE LENGTH = 1 m
($h$ = 30.5 m, $2p$ = 16.5 mm, $2e$ = 1.3 mm, $\varepsilon_p$ = 1.65, $r$ = 0.0134 ohm/m)

| | $z_1$(ohms) | | $\Delta z$(ohms) | | | z(ohm) | |
|---|---|---|---|---|---|---|---|
| f(kHz) | Re($z_1$) | I($z_1$) | Re($\Delta z$) | I($\Delta z$) | $\Delta r$ | R | X |
| 10. | .605 | 2.120 | .000 | .970 | .406 | 1.011 | 3.095 |
| 20. | .931 | 3.985 | .000 | 1.952 | .406 | 1.308 | 5.937 |
| 30. | 1.200 | 5.752 | .000 | 2.930 | .407 | 1.607 | 8.683 |
| 40. | 1.533 | 7.458 | .001 | 3.911 | .407 | 1.911 | 11.369 |
| 50. | 1.804 | 9.120 | .002 | 4.894 | .407 | 2.214 | 14.013 |
| 60. | 2.113 | 10.748 | .003 | 5.880 | .408 | 2.524 | 16.628 |
| 70. | 2.415 | 12.350 | .005 | 6.871 | .409 | 2.829 | 19.221 |
| 80. | 2.729 | 13.933 | .007 | 7.867 | .409 | 3.145 | 21.800 |
| 90. | 3.045 | 15.500 | .010 | 8.868 | .410 | 3.466 | 24.368 |
| 100. | 3.353 | 17.052 | .014 | 9.876 | .411 | 3.778 | 26.927 |
| 110. | 3.677 | 18.597 | .018 | 10.890 | .412 | 4.107 | 29.487 |
| 120. | 4.004 | 20.137 | .024 | 11.912 | .413 | 4.441 | 32.049 |
| 130. | 4.336 | 21.672 | .031 | 12.942 | .414 | 4.781 | 34.614 |
| 140. | 4.655 | 23.196 | .039 | 13.981 | .416 | 5.109 | 37.180 |
| 150. | 4.995 | 24.731 | .048 | 15.030 | .417 | 5.461 | 39.762 |
| 160. | 5.343 | 26.267 | .059 | 16.090 | .419 | 5.820 | 42.356 |
| 200. | 6.767 | 32.443 | .118 | 20.448 | .426 | 7.310 | 52.891 |
| 250. | 8.749 | 40.421 | .241 | 26.227 | .437 | 9.426 | 66.648 |
| 300. | 10.985 | 48.841 | .440 | 32.486 | .451 | 11.876 | 81.327 |

crucially affects the antenna resistance. In agreement with the physical intuition, it appears also that with the increase of frequency the effect of the contact resistance becomes less important, at 300 kHz the effect of the electrode size on the antenna resistance being negligible. Experimental data, provided by T. Susi of Naval Underwater System Center, referring to an end-grounded antenna having the same length, cable diameter, and conductor diameter, are shown in Table 5. The electrodes consist of helical wires housed in the outer layer of the foam jacket. They are of different lengths, the larger electrode being approximately 5 in. long. From a comparison with Tables 1 to 4, it appears that the contact resistance of these

TABLE 5. MEASURED IMPEDANCE OF A 30.5 m (100 ft) END-GROUNDED ANTENNA

| Frequency (kHz) | Impedance (ohms) |
|---|---|
| 10 | 3.7 + j3.4 |
| 20 | 4.0 + j6.5 |
| 30 | 4.3 + j9.5 |
| 40 | 4.6 + j12.4 |
| 50 | 5.0 + j15.3 |
| 60 | 5.3 + j18.1 |
| 70 | 5.6 + j20.9 |
| 80 | 6.0 + j23.6 |
| 90 | 6.3 + j26.4 |
| 100 | 6.7 + j29.1 |
| 110 | 7.0 + j31.8 |
| 120 | 7.4 + j34.6 |
| 130 | 7.7 + j37.3 |
| 140 | 8.1 + j40.4 |
| 150 | 8.4 + j42.7 |
| 160 | 8.8 + j45.2 |
| 200 | 10.2 + j56.2 |
| 250 | 12.2 + j69.9 |
| 300 | 14.3 + j84.1 |

electrodes is roughly equivalent to that of two cylindrical electrodes having a length of approximately 3.5 cm. The agreement with the calculated data is good. Better agreement probably would have been obtained if the variation of the conductivity with frequency had been taken into account in the computations, instead of using a constant value equal to $\sigma$ = 4.2 mho/m.

Table 6 shows the calculated impedance of an antenna having the same diameter as in Tables 1 through 4 as a function of the antenna length for several electrode lengths.

TABLE 6. ANTENNA IMPEDANCE VS. LENGTH (h) AND ELECTRODE LENGTH (L).
(f = 18 kHz, 2p = 16.5 mm, 2e = 1.3 mm, $\varepsilon_p$ = 1.65, r = 0.0134 ohm/m)

| h | L | | |
|---|---|---|---|
| | 5 cm | 30 cm | 1 m |
| 10 m | 2.9 + j1.7 ohms | 1.1 + j1.7 ohms | 0.6 + j1.8 ohms |
| 20 m | 3.2 + j3.4 ohms | 1.4 + j3.5 ohms | 0.9 + j3.5 ohms |
| 40 m | 3.5 + j6.9 ohms | 2.0 + j7.0 ohms | 1.6 + j7.0 ohms |
| 80 m | 3.4 + j13.9 ohms | 3.1 + j14.0 ohms | 2.8 + j14.0 ohms |
| 160 m | 5.2 + j28.1 ohms | 5.2 + j28.1 ohms | 5.2 + j28.2 ohms |

In Table 7, the antenna parameters are as in Table 6, except for the much smaller antenna diameter. The main effect is an increase of the resistance, the physical reason lying in the reduced area of the electrodes.

TABLE 7. ANTENNA IMPEDANCE VS. LENGTH (h) AND ELECTRODE LENGTH (L).
(f = 18 kHz, 2p = 3.2 mm, 2e = 1.3 mm, $\varepsilon_p$ = 1.65, r = 0.0134 ohm/m)

| h | L | | |
|---|---|---|---|
| | 5 cm | 30 cm | 1 m |
| 10 m | 4.9 + j1.7 ohms | 1.5 + 1.7 ohms | 0.7 + j1.8 ohms |
| 20 m | 5.2 + j3.5 ohms | 1.8 + j3.5 ohms | 1.0 + j3.6 ohms |
| 40 m | 5.4 + j7.0 ohms | 2.4 + j7.0 ohms | 1.7 + j7.1 ohms |
| 80 m | 4.5 + j14.0 ohms | 3.3 + j14.4 ohms | 2.0 + j14.2 ohms |
| 160 m | 5.3 + j28.7 ohms | 5.3 + j28.8 ohms | 5.3 + j28.9 ohms |

# 5. ANALYSIS OF THE DC CONTACT RESISTANCE OF THE ELECTRODES

In Section 4.1, a procedure has been discussed for the determination of the impedance of an electrically short, submerged antenna, grounded in the end regions. The numerical calculations of Section 4.2 show the crucial role played by the electrode geometry in determining the resistance of a VLF antenna. As expected, longer and thicker electrodes are concomitant with a lower antenna resistance.

On the basis of these numerical results, it is physically clear that the antenna resistance, neglecting the ohmic resistance of the wire, can be heuristically considered as the sum of a radiation resistance and a contact resistance. To gain better insight into the situation, the dc resistance of the structure of Fig. 6 has been evaluated. To do that, interestingly enough, nothing had to be changed in the procedure described in Section 4.1, the formulation remaining valid also for $\omega = 0$. Notice that the second term in the expression (83) now becomes identically zero. Also, in all the functions appearing in (87)

$$k_2 = 0$$

and

$$\omega \varepsilon_2 = -j\sigma \quad .$$

The current on the antenna wire, assumed here to be a perfect conductor, is evidently constant. As it should be, the computed impedance of the "dc antenna" is purely resistive. Some values are shown in Table 8 for the different antenna lengths and electrode lengths. Notice that for the shortest electrodes, there is a small difference in the computed values of the resistance for the greatest of antenna lengths considered. This is expected because the computed resistance pertains to the entire dc circuit, including both the electrodes and the current path in the seawater, different for different antenna lengths. However, because the current density is high only in the immediate neighborhood of the electrodes, the resistance is mostly localized in that region. This explains the weak dependence of the dc resistance upon the antenna length.

TABLE 8. DC RESISTANCE (2p = 16.5 mm, 2c = 1.3 mm)

| h | L | | |
|---|---|---|---|
| | 5 cm | 30 cm | 1 m |
| 10 m | 2.6 R(ohms) | 0.8 R(ohms) | 0.3 R(ohms) |
| 20 m | 2.5 R(ohms) | 0.8 R(ohms) | 0.3 R(ohms) |
| 40 m | 2.5 R(ohms) | 0.8 R(ohms) | 0.3 R(ohms) |

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# APPENDIX A

# OPEN CIRCUIT RECEIVE VOLTAGE

## APPENDIX A

## OPEN CIRCUIT RECEIVE VOLTAGE

Since the matter is well known, only a sketchy derivation of the fundamental relationship is given (for a detailed discussion see Ref. 1).

Consider the antenna terminals connected to a current open circuit generator (Fig. A-1). Let $\underline{E}_r$, $\underline{H}_r$ the receive field in which the antenna is immersed, that is, the field that would exist in the region occupied by the antenna in the absence of the antenna conductor. If $I_o$ is the strength of the current generator, the receive voltage is obtained via Lorentz's reciprocity theorem. If $\underline{E}_t$, $\underline{H}_t$ is the transmit field when the antenna is fed by the generator, the following relationship holds:

$$\iint_{o} (\underline{E}_t \times \underline{H}_r - \underline{E}_r \times \underline{H}_t) \cdot \hat{n} \, d\sigma = I_o V_r \quad , \tag{A-1}$$

where $V_r$ is the open circuit receive voltage, $\sigma_o$ is a surface completely surrounding the antenna, chosen here as shown in Fig. 7, and $\hat{n}$ is the unit vector normal to the surface. Since $\underline{E}_t$ has no circumferential component and $\underline{H}_r$ is approximately constant on a length scale equal to the antenna diameter, the first term in the integrand does not give any contribution to (A-1). Thus, (A-1) can be simplified as follows:

$$\frac{1}{I_o} \iint_{o} (\hat{n} \times \underline{H}_t) \cdot \underline{E}_r = V_r \quad . \tag{A-2}$$

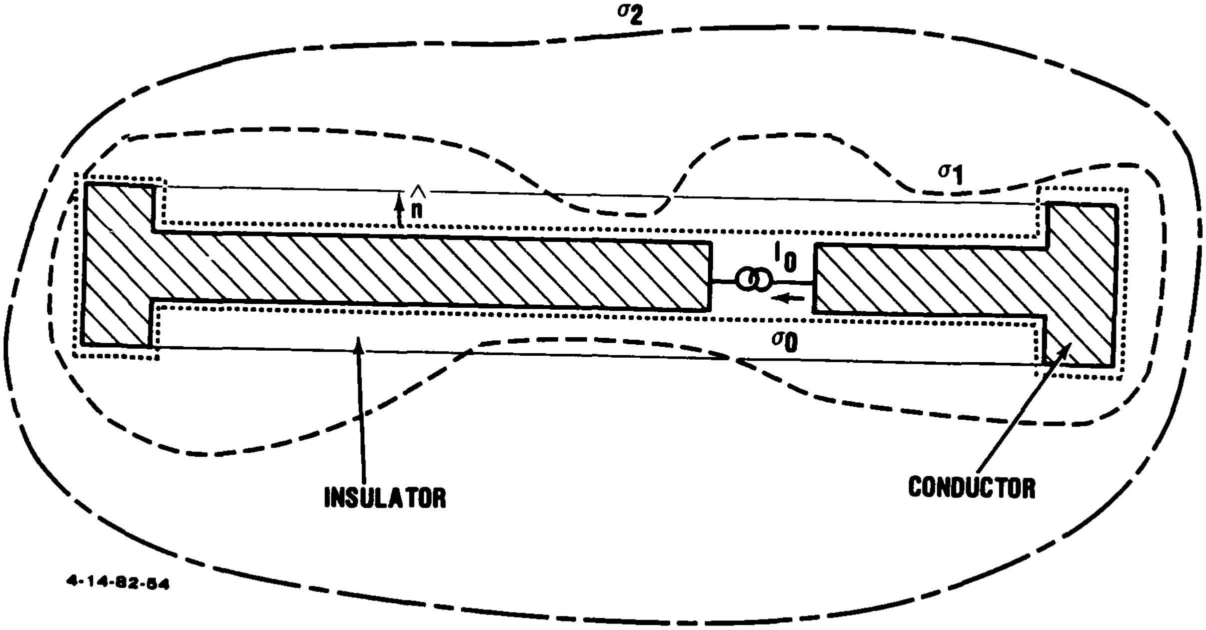


FIGURE A-1. Illustration of the equivalence principle

But with extremely good approximation

$$\underline{H}_t(s) = \frac{I(s)}{2\pi p}\,\hat{t} \quad , \qquad \text{(A-3)}$$

where s is an abscissa on the antenna and $\hat{t}$ is a circumferentially directed unit vector. Since $\underline{E}_r$ is constant on a length scale equal to the antenna diameter, from (A-2) and (A-3):

$$V_r = \frac{1}{I_o} \int_o^h E(s)I(s)ds;$$

that is, it is the same as (1) for $I_o = 1$.

# APPENDIX B

## THE ANTENNA IMPEDANCE EVALUATION AS A SCATTERING PROBLEM

## APPENDIX B

## THE ANTENNA IMPEDANCE EVALUATION AS A SCATTERING PROBLEM

Consider an antenna consisting of a metal structure, possibly coated totally or partially with dielectric, as shown in Fig. A-1. Consider a surface $\sigma$ completely enclosing the metal antenna surface. The surface can be otherwise arbitrary: examples of possible $\sigma$'s are the surfaces $\sigma_0$, $\sigma_1$, and $\sigma_2$ of Fig. A-1. Let $\underline{E}_t$, $\underline{H}_t$ denote the field supported by the antenna in transmit operation. Assume now that the system of sources inside $\sigma$ is removed. The actual sources are then replaced by a distribution of magnetic and electric currents on $\sigma$ equal to

$$M(\underline{r}_\sigma) = \underline{E}_t\ (\underline{r}_\sigma)\ x\hat{n}\ , \tag{B-1}$$

and

$$J(\underline{r}_\sigma) = \hat{n}\ x\ \underline{H}_t\ (\underline{r}_\sigma)\quad . \tag{B-2}$$

What we have done is to replace the actual electromagnetic problem with a different one with different sources. The equivalence theorem of interest here states that <u>outside $\sigma$</u> the field in the original and in the modified problem are identical. Furthermore, <u>inside $\sigma$</u> the field in the new problem is zero. Thus, the equivalent currents (B-1) and (B-2) support the actual field outside $\sigma$ (the equivalence region) but generate zero field inside $\sigma$. These statements can be rigorously proved in several ways. One of them is amazingly simple and is based on the

uniqueness theorem. It is recalled that this theorem states that the field in both the regions inside and outside $\sigma$ is uniquely determined by the tangential field at $\sigma^-$ and $\sigma^+$, respectively (rigorously one has to postulate some dissipative losses, possibly infinitesimally small, present in the media). The superscripts "+" and "-" refer to points immediately outside and inside $\sigma$, respectively. This theorem will be used in conjunction with the fact that surface currents necessarily correspond to discontinuities of the tangential field:

$$\underline{M}(\underline{r}_\sigma) = (\underline{E}_t{}^+ - \underline{E}_t{}^-) \times \hat{n} \tag{B-3}$$

and

$$\underline{J}(\underline{r}_\sigma) = \hat{n} \times (\underline{H}_t{}^+ - \underline{H}_t{}^-) \quad . \tag{B-4}$$

If the magnetic and electric currents are given by (B-1) and (B-2) at $\sigma^+$, then (B-3) and (B-4) imply that the electric and magnetic tangential field, in the new electromagnetic problem replacing the original one, are equal to $\underline{E}_t(\underline{r}_\sigma)$ and $\underline{H}_t(\underline{r}_\sigma)$ on $\sigma^+$ and are zero on $\sigma^-$. Consequently, by invoking the uniqueness theorem, we can state that the system of currents (B-1) and (B-2) support the field of the actual problem outside $\sigma$, the region of equivalence, and produce zero field in the points inside $\sigma$. The latter fact means that the field generated outside $\sigma$ by the equivalent current (B-1) and (B-2) on $\sigma^+$ is not affected by any change in the region inside $\sigma$ where the field in the equivalent problem is zero. On the basis of this observation, choose $\sigma$ as the surface of the conductor, extended, however, through the gap region, and assume that the region inside $\sigma$ is completely filled with metal. This means in the case of interest that the actual antenna structure--having an excitation gap--is replaced by an uninterrupted cylindrical conductor excited by a ribbon of magnetic current immediately

outside it. Then a (B-4) is obtained as the solution of a scattering problem: the current on $\sigma$ must be such to generate an electric tangential field in the gap region equal and opposite to that pertaining to the magnetic current (B-1).

# APPENDIX C

## DERIVATION OF (41) (42)

# APPENDIX C

## DERIVATION OF (41) (42)

Equation (42) is simply the FT of (36). From (35) and (37) one obtains

$$- jw \hat{E}_\rho - \frac{d\hat{E}z}{d\rho} = - j\omega\mu H_\phi \qquad \text{(C-1)}$$

and

$$\hat{H}_\phi \frac{w}{\omega\varepsilon} = \hat{E}_\rho \quad . \qquad \text{(C-2)}$$

Eliminating $\hat{E}_\rho$ between (C-1) and (C-2), one obtains (41).

# APPENDIX D

## ALTERNATIVE DERIVATION OF (58)

## APPENDIX D

## ALTERNATIVE DERIVATION OF (58)

As a partial check of the analysis of Section 3, (58), valid for a homogeneous medium, will be rederived via an alternative approach whose validity holds only for homogeneous media.

For the antenna in a homogeneous medium, a well-known expression for the field supported by a current distribution is the following (Ref. 9):

$$\underline{E}(\underline{r}) = -j\omega\mu \left[1 + \frac{\nabla\nabla}{k^2}\right] \iint_\sigma \underline{J}(\underline{r}_\sigma)\ \psi\ (|\underline{r}-\underline{r}_\sigma|)d\sigma\ , \qquad \text{(D-1)}$$

where $\underline{r}_\sigma$ represents a point of the antenna surface and

$$\psi(r) = \frac{e^{jkr}}{4\pi r} \qquad \text{(D-2)}$$

is the scalar Green's function for unbounded medium. More explicitly, for the tubular antenna, (D-1) is written as follows:

$$\underline{E}(z,\rho) = -\ j\omega\mu \left\{ \hat{z} + \frac{1}{k^2}\left[\hat{z}\frac{\partial}{\partial z} + \hat{\rho}\frac{\partial}{\partial \rho}\right]\frac{\partial}{\partial z}\right\} \int_{-h/2}^{h/2} dz' \int_0^{2\pi} e d\phi'$$

$$\frac{I(z')}{2\pi e}\ \psi\left(\sqrt{(z-z')^2 + \rho^2 + e^2 - 2e\rho\cos\phi'}\right)\ , \qquad \text{(D-3)}$$

which holds for arbitrary $\rho$, z. Recall the well-known integral representation for $\psi(\underline{r})$ in terms of a cylindrical function (Ref. 8, page 244)

$$\psi(\underline{r}) = \frac{1}{8\pi j}\int_{-\infty}^{+\infty} H_o^{(2)}\left(\sqrt{k^2-\lambda^2}\ \rho\right) e^{j\lambda z}\, d\lambda \ , \qquad \text{(D-4)}$$

which, inserted in (D-3) yields

$$\underline{E}(z,\rho) = -j\omega\mu \left\{ \hat{z}\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) + \hat{\rho}\,\frac{\partial^2}{\partial\rho\partial z} \right\}$$

$$\frac{1}{2\pi}\int_o^{2\pi} d\phi' \int_{-\frac{h}{2}}^{\frac{h}{2}} I(z')dz' \int_{-\infty}^{+\infty} \frac{1}{8\pi j} H_o^{(2)}$$

$$\left[\sqrt{k^2-\lambda^2}\ (\rho^2+e^2-2\rho e\cos\phi')\right] e^{j\lambda(z'-z)} d\lambda \ . \qquad \text{(D-5)}$$

The FT of the z component of (D-5) is clearly

$$\hat{E}_z(w,\rho) = -j\omega\mu \left(1 - \frac{w^2}{k^2}\right)\frac{1}{2\pi}\int_o^{2\pi} d\phi'$$

$$\cdot\ \frac{1}{4j} H_o^{(2)} \left[\sqrt{k^2-w^2}\left(\rho^2+e^2-2\rho e\cos\phi'\right)\right] \frac{1}{\sqrt{2\pi}}\int_{-\frac{h}{2}}^{\frac{h}{2}} I(z')e^{jwz'}dz' . \qquad \text{(D-6)}$$

By invoking the well-known expansion (Ref [8], page 232)

$$H_o^{(2)}\ (x+y-2xy\ \cos\phi) = \sum_{n=-\infty}^{+\infty} H_n^{(2)}(x)\ J_n^{(2)}(y)e^{jn\phi} \ , \qquad \text{(D-7)}$$

valid for $x \geq y$, and recalling the definition of $\hat{I}(w)$, one obtains

$$\hat{E}_z(w,e) = -\frac{k^2-w^2}{4\omega\varepsilon} H_o^{(2)}\left(e\sqrt{k^2-w^2}\right) J_o\left(e\sqrt{k^2-w^2}\right)\hat{I}(w) , \quad \text{(D-8)}$$

which reestablishes (58).

# APPENDIX E

# WAVE IMPEDANCE FOR A RADIALLY STRATIFIED MEDIUM

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The study of a radially stratified medium is relevant for applications other than submerged cable antennas, the case of primary concern here. Therefore, the problem will be discussed only cursorily and analytical results will be given only for the case of a multilayered jacket whose outer radius is "small" in a sense clarified below.

Label with the index "k" each region in which the permittivity is constant. Thus, if the number of layers is L, there will be L + 1 dielectric constants, the first and the last, $\varepsilon_1$ and $\varepsilon_{L+1}$, pertaining to the region in contact with the antenna conductor and to the surrounding medium, respectively. Call $\rho_k$ the radius of the cylindrical interface between the kth and the (k+1)th medium (see Fig. E-1 in which L = 3). The evaluation of $z^+(w)$ is, of course, based on the solution of (41) (42), with the appropriate radiation condition, for the case of dielectric constant variable with $\rho$ discussed here. The radiation condition is enforced by requiring that at $\rho = \rho_L$ the radius of the interface between the jacket and the medium, the wave impedance be equal to that of an outgoing cylindrical wave in a homogeneous medium having a dielectric constant equal to $\varepsilon_{L+1}$. Therefore, the wave impedance $z(w,\rho_L)$ is given by (51), evaluated for $\varepsilon = \varepsilon_L$. To obtain $z^+(w)$, the continuity of $\hat{E}_z$ and $\hat{H}_\phi$ is enforced at the various interfaces, by using well-known expressions for the functional form of the field in each layer; see for example, (Ref. [8], pages 198-216). A number of linear relationships is established which allow one

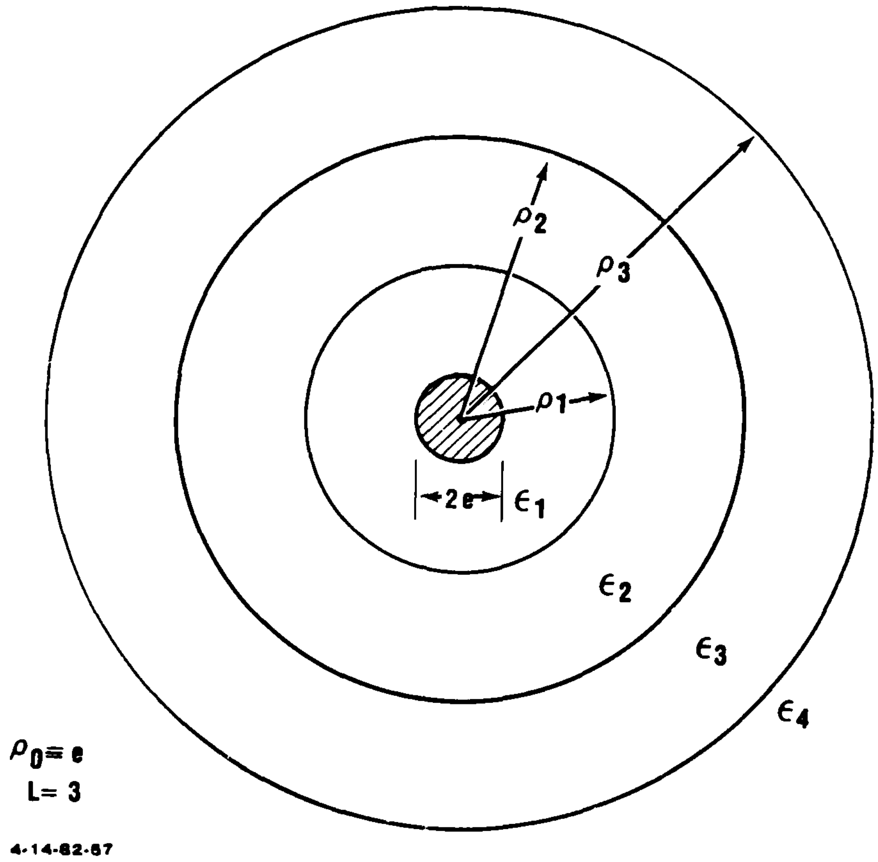


FIGURE E-1. Radially stratified medium

to eliminate the amplitudes of the wave components in each layer and to establish finally the impedance relationship sought between $\hat{E}_z$ and $\hat{H}_\phi$ at $\rho = e$. The method is straightforward, but the expression for $z^+(w)$ is involved for more than one layer. However, if one assumes that for every layer

$$|k_s \ \rho_s| << 1 \qquad (s = 1, \ldots, L) \ , \qquad \text{(E-1)}$$

an approximate expression for $z^+(w)$ is promptly established. Assume that for $\rho < \rho_L$ the relationship (67) holds. Then, by generalizing the derivation of (69) of Example 2 of Section 3, one finds

$$\hat{E}_z(w,e) = \hat{E}_z(w,\rho_L) - j\omega\mu \frac{\hat{I}(w)}{2\pi} \sum_{s=1}^{L} \frac{k_s^2 - w^2}{k_s^2} \log\left(\frac{\rho_s}{\rho_{s-1}}\right), \quad \text{(E-2)}$$

where it is understood that $\rho_o \equiv e$. From (E-2),

$$z^+(w) = z(w,\rho_L)\,\frac{e}{\rho_L} - j\omega\mu e \sum_{s=1}^{L} \frac{k_s^2 - w^2}{k_s^2} \log\left(\frac{\rho_s}{\rho_{s-1}}\right),$$

the expression sought.

# APPENDIX F

## DERIVATION OF THE EXPRESSIONS OF THE ELEMENTS OF $\underline{\underline{M}}$

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The operator L( ) appearing in (26) is defined by the right side of (60). By recalling (10), one obtains from (26)

$$m_{sk} = \frac{1}{2\pi e}\int_{0}^{2\pi} e\,d\phi \int_{-h/2}^{h/2} I_s(z)\ dz \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \hat{I}_k(w)\ z(w)\ e^{-jwz}\ dw, \quad (F\text{-}1)$$

or

$$m_{sk} = \int_{-\infty}^{+\infty} \hat{I}_k(w)\, z(w) dw \quad \frac{1}{\sqrt{2\pi}}\int_{-h/2}^{h/2} I_s(z)\ e^{-jwz}\ dz\ , \qquad (F\text{-}2)$$

which is equal to (64).

# APPENDIX G

## APPROXIMATE EXPRESSION FOR THE CURRENT IN THE END-FED GROUNDED ANTENNA AND FOR ITS FOURIER TRANSFORM

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The fundamental mode in the guiding structure whose cross section is in Fig. 10, has a propagation constant given in Ref. 7:

$$\gamma = \frac{2\pi}{\lambda_o}\sqrt{\varepsilon_p}\left\{1 - \frac{j\frac{\pi}{4} + \log\left[0.89\ p(\omega\mu\sigma)^{1/2}\right]}{\log\left(\frac{p}{e}\right)}\right\}^{1/2}, \quad \text{(G-1)}$$

where $\lambda_o$ is the wavelength in free space, $\varepsilon_p$ is the relative dielectric constant of the insulating coat, $\sigma$ is the conductivity of the medium and the hypothesis has been made that for the frequency of interest the imaginary part of the dielectric constant in the medium is much greater than the real part.

Since the origin of the abscissae on the antenna is at the feed point, the expression of the current is

$$I(x) = Ae^{-j\gamma z} - Be^{j\gamma z} . \quad \text{(G-2)}$$

With current normalization $I(0) = 1$, from (G-2),

$$A - B = 1 . \quad \text{(G-3)}$$

Because of the short-circuit condition at $z = h$,

$$Ae^{-j\gamma h} + Be^{j\gamma h} = 0 , \quad \text{(G-4)}$$

from which one finds

$$A = \frac{e^{j\gamma h}}{e^{j\gamma h} + e^{-j\gamma h}} \; ; \quad B = \frac{-e^{-j\gamma h}}{e^{j\gamma h} + e^{-j\gamma h}} \, , \tag{G-5}$$

and therefore on the antenna insulated region

$$I_c(z) = \frac{e^{j\gamma(h-z)} + e^{-j\gamma(h-z)}}{e^{j\gamma h} + e^{-j\gamma h}} \, . \tag{G-6}$$

The subscript "c" (for "conductor") indicates that the expression (66) holds on the insulated part of the antenna $0 < z \leq h$. The current on the electrode far from the feed point has values varying from $I_c(h)$ to zero, according to the discussion in Section 4. Thus, for $h < z < h + L$, where L is the electrode length:

$$I_{e1}(z) = \frac{2\, e^{j\gamma h}}{e^{j2\gamma h}+1} \left[1 - \frac{1}{L}\,(z-h)\right] \, . \tag{G-7}$$

On the other electrode, that is, for $- L < z < 0$, the current is varying from 1 (at $z = 0$) to zero (at $z = -L$). Therefore, its functional form is

$$I_{e2}(z) = 1 + \frac{z}{L} \, . \tag{G-8}$$

The FT of the current is the sum of the FTs of the currents on the antenna insulated parts:

$$I(w) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{h+L} \left[I_c(z) + I_{e1}(z) + I_{e2}(z)\right] e^{jwz} dz. \tag{G-9}$$

The evaluation of (G-9) is tedious but straightforward and the transforms of the three terms are:

$$\hat{I}_c(w) = \frac{1}{\sqrt{2\pi}} \frac{1}{1+e^{j2\gamma h}} \frac{2jwe^{j(w+\gamma)h} - j(w+\gamma)e^{j2\gamma h} - j(w-\gamma)}{\gamma^2 - w^2} . \qquad \text{(G-10)}$$

$$\hat{I}_{e1}(w) = \frac{1}{\sqrt{2\pi}} \frac{2e^{j\gamma h}e^{jw\left(h+\frac{L}{2}\right)}}{1+e^{j2\gamma h}} \frac{L}{2} \left[\frac{\sin\left(\frac{wL}{2}\right)}{\frac{wL}{2}}\right.$$

$$\left. + j \frac{\cos\left(\frac{wL}{2}\right)}{\frac{wL}{2}} - j \frac{\sin\left(\frac{wL}{2}\right)}{\left(\frac{wL}{2}\right)^2}\right] . \qquad \text{(G-11)}$$

$$\hat{I}_{e2}(w) = \frac{1}{\sqrt{2\pi}} e^{-jw\frac{L}{2}} \frac{L}{2} \left[\frac{\sin\left(\frac{wL}{2}\right)}{\frac{wL}{2}} - j \frac{\cos\left(\frac{wL}{2}\right)}{\frac{wL}{2}}\right.$$

$$\left. + j \frac{\sin\left(\frac{wL}{2}\right)}{\left(\frac{wL}{2}\right)^2}\right] . \qquad \text{(G-12)}$$

# APPENDIX H

## DERIVATION OF $\Delta Z$ FOR THE END-GROUNDED ANTENNA

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The relevant field equations in the region inside the cable are:

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = - j\omega\mu H_\phi \qquad \text{(H-1)}$$

and

$$\frac{\partial H_\phi}{\partial z} = j\omega\varepsilon_1 E_\rho \quad , \qquad \text{(H-2)}$$

with $\varepsilon_1 = \varepsilon_o \varepsilon_p$. Also, we assume that inside the cable, that is, for $\rho < p$, (66) is valid

$$H_\phi(z,\rho) = \frac{I_c(z)}{2\pi\rho} \quad . \qquad \text{(H-3)}$$

With reference to Fig. 6, the left electrode's right end is located at $z = 0^-$, that is, immediately on the left of the antenna feed point. Notice that inside the cable $E_\rho$ is zero for $z = 0^-$, and, according to (H-2), is proportional to the derivative of $I(z)$ in $0^+ < z < h$. Therefore, the left side of (H-2) has a discontinuity at $z = 0$. (Of course, this non-physical feature is a consequence of the non-physical assumption of a point-like source at $z = 0$.) Consistent with this fact, we

should properly write (H-2), for points inside the cable, as follows:

$$E_\rho = \frac{-1}{j\omega\varepsilon_1 2\pi\rho} \left[ \frac{dI_c(z)}{dz} + \frac{dI_c(z)}{dz}\bigg|_{z=0} 1(z) \right] , \qquad \text{(H-4)}$$

where 1(z) is the unit step function. The derivative of (H-4) with respect to z is

$$\frac{\partial E_\rho}{\partial z} = \frac{-1}{j\omega\varepsilon_1 2\pi\rho} \left[ \frac{d^2 I_c(z)}{dz^2} + \frac{dI_c(z)}{dz}\bigg|_{z=0} \delta(z) \right] , \qquad \text{(H-5)}$$

valid for $0 < z < h$. By inserting (H-3) and (H-5) into (H-1), one obtains

$$\frac{\partial E_z}{\partial \rho} = \frac{1}{2\pi\rho} \left\{ j\omega\mu\, I_c(z) - \frac{1}{j\omega\varepsilon_1} \left[ \frac{d^2 I_c(z)}{dz^2} + \frac{dI_c(z)}{dz}\bigg|_{z=0} \delta(z) \right] \right\} \qquad \text{(H-6)}$$

or, by integrating (H-6) with respect to $\rho$ from $\rho = e$ to $\rho = p$

$$E_z(z,p) - E_z(z,e) = \frac{1}{2\pi} \log \left(\frac{p}{e}\right) \left\{ \ldots \right\} , \qquad \text{(H-7)}$$

where the expression in curl brackets is as in (H-6). When (H-7) is inserted into the expression (85) of $\Delta z$, one obtains:

$$\Delta z = \frac{j\omega\mu}{2\pi} \log \left(\frac{p}{e}\right) \left\{ \int_0^h I_c^2(z)\, dz + \frac{1}{k_1^2} \int_0^h I_c \frac{d^2 I_c}{dz^2}\, dz \right.$$

$$\left. + \frac{1}{k_1^2} \left[ I_c \frac{dI_c}{dz} \right]_{z=0} \right\} . \qquad \text{(H-8)}$$

The second integral is evaluated by parts as follows:

$$\int_0^h I_c \frac{d^2 I_c}{dz^2}\, dz = - I_c \frac{dI_c}{dz}\bigg|_{z=0} - \int_0^h \left(\frac{dI_c}{dz}\right)^2 dz \; . \qquad \text{(H-9)}$$

where the fact that $\frac{dI_c}{dz}$ is zero at $z = h$ has been used. Thus, the first term on the right side of (H-9) cancels out with the last term in (H-8). On the other hand, the last term of (H-9) is negligible, for a short antenna, with respect to the first term of (H-8). To see this, consider the approximation (72) for the current and insert it in (H-9). We obtain

$$-\frac{1}{k_1^2}\int_0^h \left(\frac{dI_c}{dz}\right)^2 dz = \frac{-\gamma^2}{k_1^2 \cos^2 \gamma h}\left(\frac{h}{2} - \frac{\sin 2\gamma h}{4\gamma}\right) , \qquad \text{(H-10)}$$

which, neglecting terms of order higher than the first in $|\lambda h|$ and recalling that $|\gamma|$ has the same order of magnitude as $k_1$, is found to be negligible with respect to

$$\int_0^h I_c^2(z)\, dz = \frac{1}{\cos^2 h\gamma}\left(\frac{h}{2} + \frac{\sin 2\gamma h}{4\gamma}\right) . \qquad \text{(H-11)}$$

The term (H-10) has the physical interpretation of a capacitive term due to the electric reactive energy trapped in the cable, obviously, for a short, end-grounded cable, much smaller than the inductive term (H-11).